Lagrange's Interpolation Formula

Unequally spaced interpolation requires the use of the divided difference formula. It is defined as

$$f(x,x_0) = \frac{f(x)-f(x_0)}{x-x_0} \tag{1}$$

$$f(x,x_0,x_1) = \frac{f(x,x_0)-f(x_0,x_1)}{x-x_1} \tag{2}$$

$$f(x,x_0,x_1,x_2) = \frac{f(x,x_0,x_1)-f(x_0,x_1,x_2)}{x-x_2} \tag{3}$$

From equation (2), the formula can be rewritten as

$$(x-x_1)\,f(x,x_0,x_1) + f(x_0,x_1) = f(x,x_0) \qquad ,$$

and the substitution of equation (1) yields,

$$(x-x_0)(x-x_1)\,f(x,x_0,x_1) + (x-x_0)\,f(x_0,x_1) + f(x_0) = f(x) \qquad .$$

The first term is considered the remainder term as it is not in the difference table, so $f(x)$ can be expressed approximately in terms of the divided differences as

$$f(x) \approx f(x_0) + (x-x_0)\,f(x_0,x_1) + (x-x_0)(x-x_1)\,f(x_0,x_1,x_2) \qquad ,$$

a second order formula. The first order formula can be written as

$$f(x) \approx f(x_0) + (x-x_0)\,f(x_0,x_1) \qquad .$$

The above formulas are the most convenient for numerical computation when the divided differences are store in a matrix form. But actual explicit formulas can be written in terms of the sample function values.

Lagrange First Order Interpolation Formula

Given

$$f(x) = f(x_0) + (x-x_0)\frac{f(x_0)-f(x_1)}{x_0-x_1} \qquad .$$

Use simplified notations $f_0 = f(x_0)$, $f_1 = f(x_1)$, to write

$$\begin{aligned} f(x) &= f_0 + \frac{(x-x_0)}{(x_1-x_0)}(f_1-f_0) \\ &= f_0\left[\frac{(x_1-x_0)-(x-x_0)}{(x_1-x_0)}\right] + \frac{(x-x_0)}{(x_1-x_0)}f_1 \\ f(x) &= \frac{(x-x_1)}{(x_0-x_1)}f_0 + \frac{(x-x_0)}{(x_1-x_0)}f_1 \end{aligned}$$

Lagrange Second Order Interpolation Formula

Given

$$f(x) = f(x_0) + (x - x_0)\frac{f(x_0) - f(x_1)}{x_0 - x_1} + (x - x_0)(x - x_1)\frac{f(x_0, x_1) - f(x_1, x_2)}{x_0 - x_2} \qquad .$$

or

$$f(x) = f_0 + (x - x_0)\frac{f_0 - f_1}{x_0 - x_1} + \frac{(x - x_0)(x - x_1)}{x_0 - x_2}\left[\frac{f_0 - f_1}{x_0 - x_1} - \frac{f_1 - f_2}{x_1 - x_2}\right]$$

Collecting terms for f_0, f_1 and f_2, and after some tedious algebraic manipulation, the second order formula can be written as

$$f(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}\, f_0 + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}\, f_1 + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}\, f_2 \quad .$$

Lagrange N-th Order Interpolation Formula

The N-th order formula can be written in the form:

$$f(x) = f_0\delta_0(x) + f_1\delta_1(x) + \ldots + f_N\delta_N(x) \qquad ,$$

in which, $\delta_j(x)$ can be written as

$$\delta_j(x) = \frac{\prod_{i=0;i\neq j}^{N}(x - x_i)}{\prod_{i=0;i\neq j}^{N}(x_j - x_i)}$$

Each term of $\delta_j(x)$ has the required properties such that (a) $\delta_j(x_i) = 0$ when $i \neq j$ and (b) $\delta_j(x_j) = 1$. The above property ensures $f(x_j) = f_j$ and none of the other sample values (f_i, $i \neq j$) participate.