Richardson Extrapolation

Interpolation is to estimate a value between a given set of known values. Extrapolation is to use known values to project a value outside of the intended range of the previous values. Using the concept of Richardson Extrapolation, very higher order integration can be achieved using only a series of values from Trapezoidal Rule. Similarly, accurate values of derivatives could be obtained using low-order central difference derivatives.

Generally, when an approximate formula is developed, for example, the Trapezoidal Rule, the formula could be written as

\[ \int_a^b f(x) \, dx \approx \frac{h}{2} f(a) + \frac{h}{2} f(b), \]

in which \( h \) is the increment between the sample points. The approximation, however, could be replaced by an equation as

\[ \int_a^b f(x) \, dx = \frac{h}{2} f(a) + \frac{h}{2} f(b) + O(h^2). \]

The expression \( O(h^2) \) is an estimate to the error resulting from the approximation, it means the “order” of the error is of \( h^2 \). If the increment is 1/2 as large, then the error should be of the order 1/4 times smaller. Simpson’s Rule has an error term of \( O(h^4) \), therefore, if the increment is 1/2 as large, the error should be of the order 1/16 times smaller. Bode’s Rule has an error of order \( (h^6) \).

Consider an equally spaced approximate integration formula of the form:

\[ \int_a^b f(x) \, dx = \sum_i w_i f(x_i) + O(h^n). \]

Richardson extrapolation assumes the term \( O(h^n) \) could written as \( Ch^n \), in which \( C \) is a constant. The higher order terms of \( h \) are ignored. Rewrite now

\[ A_h = \sum_i w_i f(x_i), \]

and

\[ A_0 = \int_a^b f(x) \, dx. \]

\( A_h \) indicates the solution was obtained using the increment \( h \) and \( A_0 \) indicates the solution was obtained using an infinitesimally small \( h \) and that it could be considered to be the exact solution. The Richardson Extrapolation approximation could then be written as

\[ A_0 = A_h + Ch^n. \]

If another approximate solution could be obtained using an increment of 2\( h \), then \( A_0 \) could be estimated as

\[ A_0 = A_{2h} + C(2h)^n. \]
The value of $C$ can be obtained by subtracting the two above algebraic equations as

$$C = \frac{A_h - A_{2h}}{(2^n - 1)h^n}$$

Using Richardson Extrapolation, the best value can be extrapolated to be

$$A_0 = A_h + \frac{1}{(2^n - 1)}(A_h - A_{2h})$$

**Richardson Extrapolation for Trapezoidal Rule**

With an order term of $O(h^2)$, the extrapolation for a better solution is

$$A_0 = A_h + \frac{1}{3}(A_h - A_{2h})$$

**Richardson Extrapolation for Simpson’s Rule**

With an order term of $O(h^4)$, the extrapolation for a better solution is

$$A_0 = A_h + \frac{1}{15}(A_h - A_{2h})$$

**Richardson Extrapolation for Bode’s Rule**

With an order term of $O(h^6)$, the extrapolation for a better solution is

$$A_0 = A_h + \frac{1}{63}(A_h - A_{2h})$$

**Some Applications of Richardson Extrapolation**

**Trapezoidal Rule**

Using 3 sample points, $x_1$, $x_2$, $x_3$ and an increment of $h$, the estimate for $A_h$ is

$$A_h = \frac{h}{2} f(x_1) + h f(x_2) + \frac{h}{2} f(x_3)$$

Using only 2 sample points, $x_1$, $x_3$ and an increment of $2h$, the estimate for $A_{2h}$ is

$$A_{2h} = \frac{(2h)}{2} f(x_1) + \frac{(2h)}{2} f(x_3)$$
Using the Richardson Extrapolation formula for Trapezoidal Rule:

\[ A_0 = A_h + \frac{1}{3} (A_h - A_{2h}) = \frac{4}{3} A_h - \frac{1}{3} A_{2h}, \]

The best estimate for \( A_0 \) is

\[ A_0 = \frac{4}{3} \left( \frac{1}{2} f(x_1) + hf(x_2) + \frac{1}{2} f(x_3) \right) - \frac{1}{3} \left( \frac{(2h)}{2} f(x_1) + \frac{(2h)}{2} f(x_3) \right) \]

\[ = \frac{h}{3} f(x_1) + \frac{4h}{3} f(x_2) + \frac{h}{3} f(x_3). \]

The result is the Simpson’s Rule. Amazing!

**Simpson’s Rule**

Using 5 sample points, \( x_1, x_2, x_3, x_4, x_5 \), and an increment of \( h \), the estimate for \( A_h \) is

\[ A_h = \frac{h}{3} f(x_1) + \frac{4h}{3} h f(x_2) + \frac{2h}{3} h f(x_3) + \frac{4h}{3} h f(x_4) + \frac{h}{3} f(x_5). \]

Using only 3 sample points, the minimum, \( x_1, x_3, x_5 \) and an increment of \( 2h \), the estimate for \( A_{2h} \) is

\[ A_{2h} = \frac{2h}{3} f(x_1) + \frac{4(2h)}{3} h f(x_3) + \frac{2h}{3} f(x_5). \]

Using the Richardson Extrapolation formula for Simpson’s Rule:

\[ A_0 = A_h + \frac{1}{15} (A_h - A_{2h}) = \frac{16}{15} A_h - \frac{1}{15} A_{2h}, \]

The best estimate for \( A_0 \) is

\[ A_0 = \frac{16}{15} \left( \frac{h}{3} f(x_1) + \frac{4h}{3} h f(x_2) + \frac{2h}{3} h f(x_3) + \frac{4h}{3} h f(x_4) + \frac{h}{3} f(x_5) \right) \]

\[ - \frac{1}{15} \left( \frac{2h}{3} f(x_1) + \frac{4(2h)}{3} h f(x_3) + \frac{2h}{3} f(x_5) \right) \]

\[ = \frac{14h}{45} f(x_1) + \frac{64h}{45} f(x_2) + \frac{24h}{45} f(x_3) + \frac{64h}{45} f(x_4) + \frac{14h}{45} f(x_5). \]

The result is the Bode’s Rule. Amazing!

**Central Difference, First Derivative**

Using 3 sample points, \( x_-, x_0, x_1 \) and an increment of \( h \), the estimate for \( A_h \), central difference for first derivative, is

\[ A_h = \frac{f_1 - f_{-1}}{2h}. \]
Using 3 sample points, but a wider interval, \( x_{-2}, x_0, x_2 \) and an increment of \( 2h \), the estimate for \( A_{2h} \) is

\[
A_{2h} = \frac{f_2 - f_{-2}}{2(2h)} .
\]

Using the Richardson Extrapolation formula for \( O(h^2) \):

\[
A_0 = A_h + \frac{1}{3} (A_h - A_{2h}) = \frac{4}{3} A_h - \frac{1}{3} A_{2h} ,
\]

The best estimate for \( A_0 \) is

\[
A_0 = \frac{4}{3} \left( \frac{f_1 - f_{-1}}{2h} \right) - \frac{1}{3} \left( \frac{f_2 - f_{-2}}{2(2h)} \right) = \frac{-f_2 + 8f_1 - 8f_{-1} + f_{-2}}{12h} .
\]

The result is that of a 4-th order polynomial fitted to 5 sample points, at \( x_{-2}, x_{-1}, x_0, x_1, x_2 \).

**Central Difference, Second Derivative**

Using 3 sample points, \( x_{-1}, x_0, x_1 \) and an increment of \( h \), the estimate for \( A_h \), central difference for second derivative, is

\[
A_h = \frac{f_1 - 2f_0 + f_{-1}}{h^2} .
\]

Using 3 sample points, but a wider interval, \( x_{-2}, x_0, x_2 \) and an increment of \( 2h \), the estimate for \( A_{2h} \) is

\[
A_{2h} = \frac{f_2 - 2f_0 + f_{-2}}{(2h)^2} .
\]

Using the Richardson Extrapolation formula for \( O(h^2) \):

\[
A_0 = A_h + \frac{1}{3} (A_h - A_{2h}) = \frac{4}{3} A_h - \frac{1}{3} A_{2h} ,
\]

The best estimate for \( A_0 \) is

\[
A_0 = \frac{4}{3} \left( \frac{f_1 - 2f_0 + f_{-1}}{h^2} \right) - \frac{1}{3} \left( \frac{f_2 - 2f_0 + f_{-2}}{4h^2} \right) = \frac{-f_2 + 16f_1 - 30f_0 + 16f_{-1} - f_{-2}}{12h^2} .
\]

The result is that of a 4-th order polynomial fitted to 5 sample points, at \( x_{-2}, x_{-1}, x_0, x_1, x_2 \).