

Rigid Body Kinetics

All formula which are applicable to a system of particles will also be applicable to Rigid Bodies. The additional constraint placed on the system of particles, which requires the distances between particles be constant, simplifies the problem substantially.

Recall the equation of motion for a system of particles is

$$\sum \vec{F} = \frac{d}{dt} \vec{L} = \frac{d}{dt} \left(\sum_{i=1}^N m_i \vec{v}_i \right) . \quad (1)$$

But for a rigid body, one additional constraint can be placed on the velocities, i.e.,

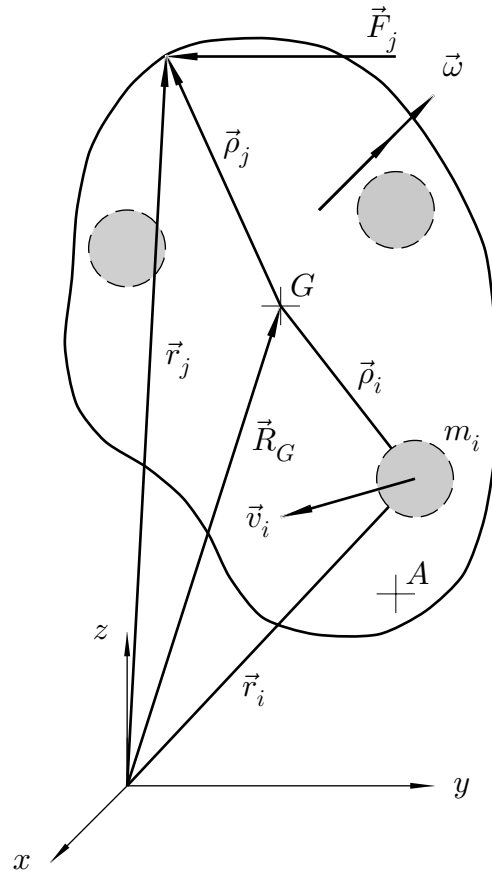
$$\begin{aligned} \vec{v}_i &= \vec{v}_G + \vec{\omega} \times \vec{r}_{i/G} \\ &= \vec{v}_G + \vec{\omega} \times \vec{\rho}_i \end{aligned} \quad (2)$$

where $\vec{\rho}_i = \vec{r}_{i/G}$ is the position vector of mass m_i with respect to the center of gravity G . Combining equations (1) and (2), we can write

$$\sum \vec{F} = \frac{d}{dt} \left(\sum_{i=1}^N m_i \vec{v}_G + \sum_{i=1}^N m_i \vec{\omega} \times \vec{\rho}_i \right) . \quad (3)$$

Removing now constants from the summation over index i , we have

$$\sum \vec{F} = \frac{d}{dt} \left(\vec{v}_G \sum_{i=1}^N m_i + \vec{\omega} \times \sum_{i=1}^N m_i \vec{\rho}_i \right) . \quad (4)$$



Define now the total mass,

$$M = \sum_{i=1}^N m_i ,$$

and recognize that

$$\sum_{i=1}^N m_i \vec{\rho}_i = \vec{0}$$

by the definition of the center of gravity, the equation of motion for rigid bodies in translation can be written simply as

$$\sum \vec{F} = \frac{d}{dt} (M \vec{v}_g) = M \vec{a}_G$$

which is identical to that for a system of particles.

Moment and Angular Momentum of a Rigid Body

Recall for a system of particles, the alternate form of Newton's 2nd Law, that of moment and angular momentum, can be written about the origin O as

$$\sum \vec{M}_O = \frac{d}{dt} \vec{H}_O \quad , \quad (1)$$

where the total moment about point O for all external applied forces is defined as

$$\sum \vec{M}_O = \sum_j \vec{r}_j \times \vec{F}_j \quad (2)$$

and the total angular momentum about point O for N mass particles is defined as

$$\vec{H}_O = \sum_{i=1}^N \vec{r}_i \times m_i \vec{v}_i \quad . \quad (3)$$

For a rigid body, the additional constraint on the velocities is such that

$$\vec{v}_i = \vec{v}_A + \vec{\omega} \times \vec{r}_{i/A} \quad , \quad (4)$$

in which A is a point on the rigid body where the velocity is known. Since Newton's Second Law for translations utilizes the velocity and acceleration of the center of gravity G , it is convenient to simplify equation (3) by substituting point G as point A in equation (4), i.e.,

$$\vec{v}_i = \vec{v}_G + \vec{\omega} \times \vec{r}_{i/G} = \vec{v}_G + \vec{\omega} \times \vec{\rho}_i \quad , \quad (5)$$

in which

$$\vec{r}_{i/G} = \vec{r}_i - \vec{R}_G = \vec{\rho}_i \quad .$$

To obtain \vec{r}_i for equation (3), rewrite the above equation as

$$\vec{r}_i = \vec{R}_G + \vec{\rho}_i \quad (6)$$

and then substitute equations (5) and (6) into equation (3) to yield

$$\begin{aligned} \vec{H}_O &= \sum_{i=1}^N (\vec{R}_G + \vec{\rho}_i) \times m_i (\vec{v}_G + \vec{\omega} \times \vec{\rho}_i) \\ &= \sum_{i=1}^N \vec{R}_G \times m_i \vec{v}_G + \sum_{i=1}^N \vec{\rho}_i \times m_i \vec{v}_G \\ &\quad + \sum_{i=1}^N \vec{R}_G \times m_i (\vec{\omega} \times \vec{\rho}_i) + \sum_{i=1}^N \vec{\rho}_i \times m_i (\vec{\omega} \times \vec{\rho}_i) \\ &= \vec{R}_G \times \left(\sum_{i=1}^N m_i \right) \vec{v}_G + \left(\sum_{i=1}^N m_i \vec{\rho}_i \right) \times \vec{v}_G \\ &\quad + \vec{R}_G \times \vec{\omega} \times \left(\sum_{i=1}^N m_i \vec{\rho}_i \right) + \sum_{i=1}^N m_i (\vec{\rho}_i \times \vec{\omega} \times \vec{\rho}_i) \quad . \end{aligned}$$

Using again the definition of the center of gravity, i.e., $\sum m_i \vec{\rho}_i = \vec{0}$, and let M be the total mass $\sum m_i$, we can write the angular momentum about the origin O as

$$\vec{H}_O = \vec{R}_G \times M \vec{v}_G + \sum_{i=1}^N m_i (\vec{\rho}_i \times \vec{\omega} \times \vec{\rho}_i) \quad . \quad (7)$$

The above expression is written in *abstract algebra* form, which implies it is applicable for any coordinate system.

The Moment of Inertia Matrix of a Rigid Body

To make equation (7) easier to apply numerically, the second term on the right-hand-side,

$$\sum_{i=1}^N m_i (\vec{\rho}_i \times \vec{\omega} \times \vec{\rho}_i) \quad ,$$

can be expressed as the matrix product $[I_G] \vec{\omega}$, thus separating completely the geometrical information contained in $\vec{\rho}$ from the kinematic information contained in $\vec{\omega}$. Otherwise, the entire summation has to be recalculated every instant the angular velocity $\vec{\omega}$ of the body changes.

In vector algebra, there is an identity for any arbitrary vectors \vec{A} , \vec{B} and \vec{C} such that

$$(\vec{A} \times \vec{B}) \times \vec{C} = (\vec{A} \cdot \vec{C}) \vec{B} - \vec{A} (\vec{B} \cdot \vec{C}) \quad . \quad (8)$$

In our application, we shall let \vec{A} and \vec{C} represent $\vec{\rho}_i$ and \vec{B} represents $\vec{\omega}$, then

$$\vec{\rho}_i \times \vec{\omega} \times \vec{\rho}_i = (\vec{\rho}_i \cdot \vec{\rho}_i) \vec{\omega} - \vec{\rho}_i (\vec{\rho}_i \cdot \vec{\omega}) \quad . \quad (9)$$

In matrix form, equation (9) can be written as

$$\vec{\rho}_i \times \vec{\omega} \times \vec{\rho}_i = (\{\rho_i\}^T \{\rho_i\}) \{\omega\} - \{\rho_i\} (\{\rho_i\}^T \{\omega\}) \quad (10)$$

or by altering the order of matrix multiplication

$$\begin{aligned} \vec{\rho}_i \times \vec{\omega} \times \vec{\rho}_i &= (\{\rho_i\}^T \{\rho_i\}) \{\omega\} - (\{\rho_i\} \{\rho_i\}^T) \{\omega\} \\ &= (\{\rho_i\}^T \{\rho_i\} [1] - \{\rho_i\} \{\rho_i\}^T) \{\omega\} \end{aligned} \quad (11)$$

where $[1]$ is the 3×3 identity matrix.

Using cartesian coordinates, let

$$\{\rho_i\} = \begin{Bmatrix} \rho_{xi} \\ \rho_{yi} \\ \rho_{zi} \end{Bmatrix} \quad \text{and} \quad \vec{\omega} = \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix}$$

then

$$\begin{aligned} \{\rho_i\}^T \{\rho_i\} &= |\vec{\rho}_i|^2 = [\rho_{xi} \quad \rho_{yi} \quad \rho_{zi}] \begin{bmatrix} \rho_{xi} \\ \rho_{yi} \\ \rho_{zi} \end{bmatrix} = (\rho_{xi}^2 + \rho_{yi}^2 + \rho_{zi}^2) \quad , \\ \{\rho_i\} \{\rho_i\}^T &= \begin{bmatrix} \rho_{xi} \\ \rho_{yi} \\ \rho_{zi} \end{bmatrix} [\rho_{xi} \quad \rho_{yi} \quad \rho_{zi}] = \begin{bmatrix} \rho_{xi}^2 & \rho_{xi} \rho_{yi} & \rho_{xi} \rho_{zi} \\ \rho_{yi} \rho_{xi} & \rho_{yi}^2 & \rho_{yi} \rho_{zi} \\ \rho_{zi} \rho_{xi} & \rho_{zi} \rho_{yi} & \rho_{zi}^2 \end{bmatrix} \quad , \end{aligned}$$

and

$$\vec{\rho}_i \times \vec{\omega} \times \vec{\rho}_i = \left((\rho_{xi}^2 + \rho_{yi}^2 + \rho_{zi}^2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \rho_{xi}^2 & \rho_{xi}\rho_{yi} & \rho_{xi}\rho_{zi} \\ \rho_{yi}\rho_{xi} & \rho_{yi}^2 & \rho_{yi}\rho_{zi} \\ \rho_{zi}\rho_{xi} & \rho_{zi}\rho_{yi} & \rho_{zi}^2 \end{bmatrix} \right) \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} \quad (12)$$

Performing now the summation over i , the moment of inertia matrix $[I_G]$ can be expressed as

$$\begin{aligned} [I_G]\vec{\omega} &= \sum_{i=1}^N m_i (\vec{\rho}_i \times \vec{\omega} \times \vec{\rho}_i) \\ &= \left(\sum_{i=1}^N m_i \begin{bmatrix} \rho_{yi}^2 + \rho_{zi}^2 & -\rho_{xi}\rho_{yi} & -\rho_{xi}\rho_{zi} \\ -\rho_{yi}\rho_{xi} & \rho_{xi}^2 + \rho_{zi}^2 & -\rho_{yi}\rho_{zi} \\ -\rho_{zi}\rho_{xi} & -\rho_{zi}\rho_{yi} & \rho_{xi}^2 + \rho_{yi}^2 \end{bmatrix} \right) \vec{\omega} \end{aligned} \quad (13)$$

Convenient Points to Formulate the Rotational Equation of Motion

As shown in equation (7), the angular momentum \vec{H}_O is dependent of both the linear momentum term, $M\vec{v}_G$, and the angular momentum term $[I_G]\vec{\omega}$. Therefore, the linear momentum equation and the angular momentum equation is normally coupled. There are two special cases, however, where the two equations of motion are uncoupled.

Case 1: Origin O is the Center of Gravity G

If the center of gravity G is the origin O , i.e., $\vec{r}_G = \vec{r}_O$, then the position vector

$$\vec{R}_G = \vec{r}_{G/O} = \vec{r}_G - \vec{r}_O = \vec{0} \quad .$$

It is clear that the first term of equation (7) vanishes and

$$\vec{H}_O = \vec{H}_G = [I_G]\vec{\omega} \quad .$$

Case 2: Origin O is a Fixed Point

If the origin O is a fixed point, then $\vec{v}_O = \vec{0}$ and

$$\vec{v}_G = \vec{v}_O + \vec{\omega} \times \vec{r}_{G/O} = \vec{\omega} \times \vec{R}_G \quad . \quad (14)$$

Substituting equation (14) into equation (7), the angular momentum \vec{H}_O can be expressed as

$$\vec{H}_O = M\vec{R}_G \times \vec{\omega} \times \vec{R}_G + \sum_{i=1}^N m_i (\vec{\rho}_i \times \vec{\omega} \times \vec{\rho}_i) \quad . \quad (15)$$

Judging from the similarity of the two terms on the right-hand-side, we can write

$$\vec{H}_O = [I_O]\vec{\omega} \quad (16)$$

where $[I_O] = [I_T] + [I_G]$ with $[I_G]$ defined by equation (13) and $[I_T]$ defined by

$$\begin{aligned} [I_T]\vec{\omega} &= M(\vec{R}_G \times \vec{\omega} \times \vec{R}_G) \\ &= M \begin{bmatrix} R_y^2 + R_z^2 & -R_x R_y & -R_x R_z \\ -R_y R_x & R_x^2 + R_z^2 & -R_y R_z \\ -R_z R_x & -R_z R_y & R_x^2 + R_y^2 \end{bmatrix} \vec{\omega} \end{aligned} \quad (17)$$

Equation (17) is the “parallel-axis theorem” in three dimensions. The values R_x , R_y and R_z are the x , y and z components of the vector \vec{R}_G , respectively. Once the moment of inertia matrix $[I_G]$ is known at the center of gravity G , then it can be translated elsewhere by first calculating and then adding the translation matrix $[I_T]$.

If for some reason the moment of inertia matrix $[I_A]$ is known at point A and it is desirable to obtain the moment of inertia matrix $[I_B]$ at point B . Then the steps which must be taken are

- (1) calculate $[I_{T1}]$ using the position vector $\vec{r}_{A/G}$,
- (2) determine $[I_G] = [I_A] - [I_{T1}]$,
- (3) calculate $[I_{T2}]$ using the position vector $\vec{r}_{B/G}$, and
- (4) obtain $[I_B] = [I_G] + [I_{T2}]$.

It is important that one of the points used during the translation of axes be the point G , it is *incorrect* to calculate $[I_B] = [I_A] + [I_T]$ in which $[I_T]$ is determined using position vector $\vec{r}_{B/A}$.

The Moment of Inertia Matrix of a System of Rigid Bodies

A complex machinery is often composed of several components which can be considered rigid bodies. Unlike particles, which are concentrated masses, each rigid-body component has its own moment of inertia matrix.

To calculate the total moment of inertia matrix for a system of N rigid bodies, assume the moment of inertia matrix at the center of mass of each rigid body $[I_{G_i}]$ is already known, then the total moment of inertia matrix of the system about point O is

$$[I_O] = \sum_{i=1}^N ([I_{G_i}] + [I_{T_i}]) \quad ,$$

in which $[I_{T_i}]$ is the translation matrix calculated using the position vector \vec{R}_{G_i} of the i -th rigid-body component.

If the components have simple geometries such as a slender rod, a thin plate, a cylinder or a sphere, then their moment of inertia matrices about their centers of mass can be formed by simply consulting a reference table. The moment of inertia matrix for a symmetric body is always diagonal.

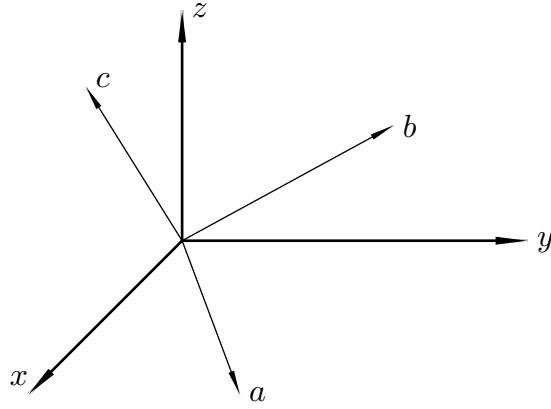
Rotational Transformation of Inertia Matrices

If the moment of inertia matrix $[I_O]_{xyz}$ is known in the xyz coordinate system, it is sometimes important to have it expressed in the abc coordinate system.

Consider first the definition of $[I_O]_{xyz}$,

$$\{H_O\}_{xyz} = [I_O]_{xyz} \{\omega\}_{xyz} \quad (1)$$

in which $\{H_O\}_{xyz}$ and $\{\omega\}_{xyz}$ are the angular momentum and the angular velocity vectors expressed in the xyz system, respectively.



Now if we define a coordinate transformation matrix $[Q]$ such that

$$\{q\}_{abc} = [Q] \{q\}_{xyz} \quad (2)$$

then for our vectors of interest, $\{H_O\}$ and $\{\omega\}$, we have

$$\{H_O\}_{abc} = [Q] \{H_O\}_{xyz} \quad , \quad (3)$$

and

$$\{\omega\}_{abc} = [Q] \{\omega\}_{xyz} \quad (4)$$

or

$$\{\omega\}_{xyz} = [Q]^T \{\omega\}_{abc} \quad . \quad (5)$$

Substitution of equations (1) and (5) into equation (3) yields

$$\{H_O\}_{abc} = [Q] \{H_O\}_{xyz} = [Q][I_O]_{xyz} \{\omega\}_{xyz} = [Q][I_O]_{xyz}[Q]^T \{\omega\}_{abc} \quad ,$$

or the equation

$$\{H_O\}_{abc} = [Q][I_O]_{xyz}[Q]^T \{\omega\}_{abc} \quad . \quad (6)$$

Define now the new relationship

$$\{H_O\}_{abc} = [I_O]_{abc} \{\omega\}_{abc} \quad . \quad (7)$$

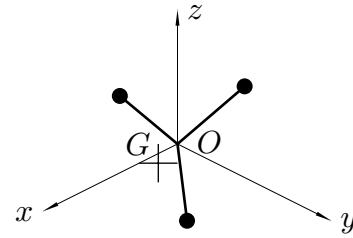
It is clear by matching equations (7) and (6) that

$$[I_O]_{abc} = [Q][I_O]_{xyz}[Q]^T \quad . \quad (8)$$

In equation (8), the transformation $[Q]$ must be defined as shown in equation (2), i.e., from xyz to abc . It is interesting to point out that two matrix multiplications are required to transform $[I_O]$ because it is a tensor of second rank.

Example – Calculation of the Moment of Inertia Matrix

Find the moment of inertia matrices $[I_O]$ and $[I_G]$ for the object represented by 3 concentrated masses with massless rods. The masses and their respective position vectors are: $m_1 = 4 \text{ kg}$, $m_2 = 5 \text{ kg}$, $m_3 = 3 \text{ kg}$ and $\vec{r}_1 = \{4, 1, -2\}^T$, $\vec{r}_2 = \{-3, 0, 3\}^T$, $\vec{r}_3 = \{2, 1, 1\}^T$.



Solution:

Since the moment of inertia matrices of concentrated masses are zero, the moment of inertia of the system can be calculated by simply adding the translation matrices using the parallel-axis theorem.

$$\begin{aligned}
 [I_O] &= 4 \left(21[1] - \begin{bmatrix} 16 & 4 & -8 \\ 4 & 1 & -2 \\ -8 & -2 & 4 \end{bmatrix} \right) + 5 \left(18[1] - \begin{bmatrix} 9 & 0 & -9 \\ 0 & 0 & 0 \\ -9 & 0 & 9 \end{bmatrix} \right) + 3 \left(6[1] - \begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \right) \\
 &= 4 \begin{bmatrix} 5 & -4 & 8 \\ -4 & 20 & 2 \\ 8 & 2 & 17 \end{bmatrix} + 5 \begin{bmatrix} 9 & 0 & 9 \\ 0 & 18 & 0 \\ 9 & 0 & 9 \end{bmatrix} + 3 \begin{bmatrix} 2 & -2 & -2 \\ -2 & 5 & -1 \\ -2 & -1 & 5 \end{bmatrix} \\
 &= \begin{bmatrix} 20 & -16 & 32 \\ -16 & 80 & 8 \\ 32 & 8 & 68 \end{bmatrix} + \begin{bmatrix} 45 & 0 & 45 \\ 0 & 90 & 0 \\ 45 & 0 & 45 \end{bmatrix} + \begin{bmatrix} 6 & -6 & -6 \\ -6 & 15 & -3 \\ -6 & -3 & 15 \end{bmatrix} = \begin{bmatrix} 71 & -22 & 71 \\ -22 & 185 & 5 \\ 71 & 5 & 128 \end{bmatrix}
 \end{aligned}$$

To find the location of the center of mass for the system of 3 particles, use the definition as

$$(4 + 5 + 3)\vec{R}_G = 4 \begin{Bmatrix} 4 \\ 1 \\ -2 \end{Bmatrix} + 5 \begin{Bmatrix} -3 \\ 0 \\ 3 \end{Bmatrix} + 3 \begin{Bmatrix} 2 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 7 \\ 7 \\ 10 \end{Bmatrix} \implies \vec{R}_G = \begin{Bmatrix} 0.583 \\ 0.583 \\ 0.833 \end{Bmatrix}$$

To find $[I_G]$ for the system of 3 concentrated masses, find the translation matrix $[I_T]$ for the entire assembly of 3 particles from point G to point O using the position vector \vec{R}_G :

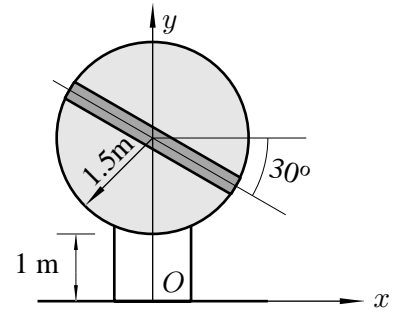
$$\begin{aligned}
 [I_T] &= 12 \left((0.583^2 + 0.583^2 + 0.833^2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.34 & 0.34 & 0.49 \\ 0.34 & 0.34 & 0.49 \\ 0.49 & 0.49 & 0.69 \end{bmatrix} \right) \\
 &= 12 \begin{bmatrix} 1.034 & -0.34 & -0.49 \\ -0.34 & 1.034 & -0.49 \\ -0.49 & -0.49 & 0.684 \end{bmatrix} = \begin{bmatrix} 12.4 & -4.1 & -5.9 \\ -4.1 & 12.4 & -5.9 \\ -5.9 & -5.9 & 8.21 \end{bmatrix} .
 \end{aligned}$$

Now, $[I_O] = [I_G] + [I_T]$ implies $[I_G] = [I_O] - [I_T]$, or

$$[I_G] = \begin{bmatrix} 58.6 & -17.9 & 76.9 \\ -17.9 & 172.6 & 10.9 \\ 76.9 & 10.9 & 119.8 \end{bmatrix}$$

Example – Moment of Inertia of a System of Rigid Bodies

An out-of-this-world object is composed of (i) a solid sphere of mass 20 kg and radius 1.5 m placed on top of (ii) a 1-meter tall massless column with (iii) a thin ring of 5 kg mass welded on the sphere's surface at an angle of 30° measured with respect to the equator. Find the 3×3 moment of inertia matrix $[I_O]$ about point O . The radius of the ring is approximately 1.5 m.



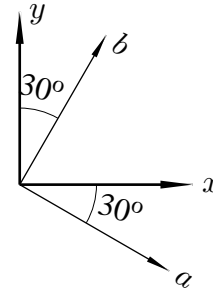
SOLUTION:

For the sphere, the moment of inertia matrix about its center of mass is

$$[I_{G1}] = \frac{2}{5}mR^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \frac{2}{5}(20)(1.5)^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{bmatrix}_{xyz}$$

For the thin ring, use system abc so that

$$[I_{G2}]_{abc} = mR^2 \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} = \begin{bmatrix} 5.625 & 0 & 0 \\ 0 & 11.25 & 0 \\ 0 & 0 & 5.625 \end{bmatrix}_{abc}$$



Define now the transformation from abc to xyz as

$$\begin{Bmatrix} q_x \\ q_y \\ q_z \end{Bmatrix} = \begin{bmatrix} \cos 30^\circ & \cos 60^\circ & 0 \\ \cos 120^\circ & \cos 30^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} q_a \\ q_b \\ q_c \end{Bmatrix} = \begin{bmatrix} 0.866 & 0.5 & 0 \\ -0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} q_a \\ q_b \\ q_c \end{Bmatrix} = [Q] \begin{Bmatrix} q_a \\ q_b \\ q_c \end{Bmatrix}$$

$$[I_{G2}]_{xyz} = [Q] \begin{bmatrix} 5.625 & 0 & 0 \\ 0 & 11.25 & 0 \\ 0 & 0 & 5.625 \end{bmatrix}_{abc} \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.866 & 0.5 & 0 \\ -0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4.87 & -2.81 & 0 \\ 5.63 & 9.74 & 0 \\ 0 & 0 & 5.63 \end{bmatrix} = \begin{bmatrix} 7.03 & 2.44 & 0 \\ 2.44 & 9.84 & 0 \\ 0 & 0 & 5.63 \end{bmatrix}_{xyz}$$

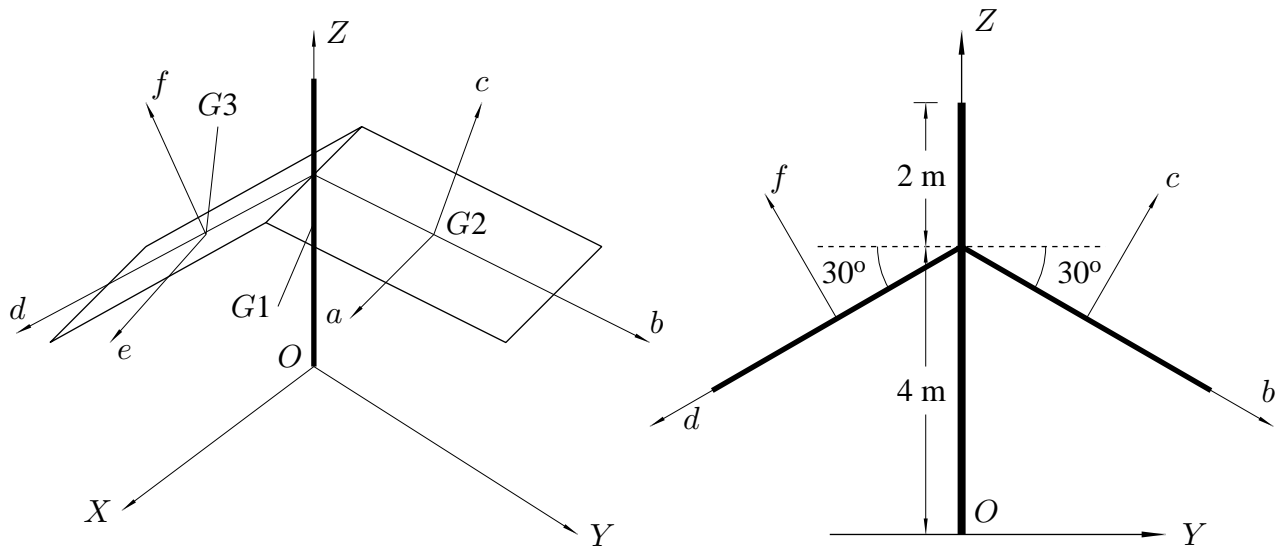
Since $\vec{R}_{G1} = \vec{R}_{G2} = \{0, 2.5, 0\}^T$

$$[I_{T1}] + [I_{T2}] = (20 + 5) \left(6.25 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 6.25 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 156.25 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 156.25 \end{bmatrix}$$

$$[I_O] = [I_{G1}] + [I_{G2}] + [I_{T1}] + [I_{T2}] = \begin{bmatrix} 181.28 & 2.44 & 0 \\ 2.44 & 27.84 & 0 \\ 0 & 0 & 179.88 \end{bmatrix}_{xyz}$$

Example – Computation of the Moment of Inertia Matrix

Find the moment of inertia matrices $[I_O]$ for the object represented by a slender rod ($m_1 = 10$ kg) and 2 slanted thin rectangular plates ($m_2 = m_3 = 5$ kg). The thin plates have the dimensions of $4\text{ m} \times 2\text{ m}$ and they are mounted onto the slender rod at an angle of 30° below the horizontal axis. (Note: the abc and def coordinate systems are specified for intermediate calculations only.)



Moment of Inertia Matrices About Respective Centroids:

$$[I_{G1}] = \begin{bmatrix} \frac{1}{12}(10)(6)^2 & 0 & 0 \\ 0 & \frac{1}{12}(10)(6)^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 30 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{XYZ}$$

$$[I_{G2}] = \begin{bmatrix} \frac{1}{12}(5)(4)^2 & 0 & 0 \\ 0 & \frac{1}{12}(5)(2)^2 & 0 \\ 0 & 0 & \frac{1}{12}(5)(4^2 + 2^2) \end{bmatrix}_{abc} = \begin{bmatrix} 6.67 & 0 & 0 \\ 0 & 1.67 & 0 \\ 0 & 0 & 8.33 \end{bmatrix}_{abc}$$

$$[I_{G3}] = \begin{bmatrix} \frac{1}{12}(5)(2)^2 & 0 & 0 \\ 0 & \frac{1}{12}(5)(4)^2 & 0 \\ 0 & 0 & \frac{1}{12}(5)(2^2 + 4^2) \end{bmatrix}_{def} = \begin{bmatrix} 1.67 & 0 & 0 \\ 0 & 6.67 & 0 \\ 0 & 0 & 8.33 \end{bmatrix}_{def}$$

Rotation Matrices

$$\begin{Bmatrix} q_X \\ q_Y \\ q_Z \end{Bmatrix} = [Q_1] \begin{Bmatrix} q_a \\ q_b \\ q_c \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 30^\circ & \cos 60^\circ \\ 0 & \cos 120^\circ & \cos 30^\circ \end{bmatrix} \begin{Bmatrix} q_a \\ q_b \\ q_c \end{Bmatrix}$$

$$\begin{Bmatrix} q_X \\ q_Y \\ q_Z \end{Bmatrix} = [Q_2] \begin{Bmatrix} q_d \\ q_e \\ q_f \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \cos 210^\circ & 0 & \cos 120^\circ \\ \cos 120^\circ & 0 & \cos 30^\circ \end{bmatrix} \begin{Bmatrix} q_d \\ q_e \\ q_f \end{Bmatrix}$$

Rotational Transformation for the Rectangular Plates

$$\begin{aligned}
 [I_{G2}]_{XYZ} &= [Q_1][I_{G2}]_{abc}[Q_1]^T = [Q_1] \begin{bmatrix} 6.67 & 0 & 0 \\ 0 & 1.67 & 0 \\ 0 & 0 & 8.33 \end{bmatrix}_{abc} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.866 & -0.5 \\ 0 & 0.5 & 0.866 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.866 & 0.5 \\ 0 & -0.5 & 0.866 \end{bmatrix} \begin{bmatrix} 6.67 & 0 & 0 \\ 0 & 1.446 & -0.835 \\ 0 & 4.165 & 7.214 \end{bmatrix} = \begin{bmatrix} 6.67 & 0 & 0 \\ 0 & 3.335 & 2.884 \\ 0 & 2.884 & 6.665 \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 [I_{G3}]_{XYZ} &= [Q_2][I_{G3}]_{def}[Q_2]^T = [Q_2] \begin{bmatrix} 1.67 & 0 & 0 \\ 0 & 6.67 & 0 \\ 0 & 0 & 8.33 \end{bmatrix}_{def} \begin{bmatrix} 0 & -0.866 & -0.5 \\ 1 & 0 & 0 \\ 0 & -0.5 & 0.866 \end{bmatrix} \\
 &= \begin{bmatrix} 0 & 1 & 0 \\ -0.866 & 0 & -0.5 \\ -0.5 & 0 & 0.866 \end{bmatrix} \begin{bmatrix} 0 & -1.446 & -0.835 \\ 6.67 & 0 & 0 \\ 0 & -4.165 & 7.214 \end{bmatrix} = \begin{bmatrix} 6.67 & 0 & 0 \\ 0 & 3.335 & -2.884 \\ 0 & -2.884 & 6.665 \end{bmatrix}
 \end{aligned}$$

By combining the three inertia matrices, we have

$$\sum_{i=1}^3 [I_{Gi}]_{XYZ} = \begin{bmatrix} 43.34 & 0 & 0 \\ 0 & 36.67 & 0 \\ 0 & 0 & 13.33 \end{bmatrix}_{XYZ} .$$

Translational Transformations for All Three Elements

$$\vec{R}_{G1} = \begin{Bmatrix} 0 \\ 0 \\ 3 \end{Bmatrix} \implies [I_{T1}] = (10) \left(\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 9 \end{bmatrix} \right) = \begin{bmatrix} 90 & 0 & 0 \\ 0 & 90 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned}
 \vec{R}_{G2} &= \begin{Bmatrix} 0 \\ 2 \cos 30^\circ \\ 4 - 2 \sin 30^\circ \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1.732 \\ 3 \end{Bmatrix} \implies \\
 [I_{T2}] &= (5) \left(\begin{bmatrix} 12 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 12 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 5.196 \\ 0 & 5.196 & 9 \end{bmatrix} \right) = \begin{bmatrix} 60 & 0 & 0 \\ 0 & 45 & -25.98 \\ 0 & -25.98 & 15 \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \vec{R}_{G3} &= \begin{Bmatrix} 0 \\ -2 \cos 30^\circ \\ 4 - 2 \sin 30^\circ \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1.732 \\ 3 \end{Bmatrix} \implies \\
 [I_{T3}] &= (5) \left(\begin{bmatrix} 12 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 12 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & -5.196 \\ 0 & -5.196 & 9 \end{bmatrix} \right) = \begin{bmatrix} 60 & 0 & 0 \\ 0 & 45 & 25.98 \\ 0 & 25.98 & 15 \end{bmatrix}
 \end{aligned}$$

By combining the three translation matrices (parallel-axis theorem), we have

$$\sum_{i=1}^3 [I_{Ti}]_{XYZ} = \begin{bmatrix} 210 & 0 & 0 \\ 0 & 180 & 0 \\ 0 & 0 & 30 \end{bmatrix}_{XYZ} ,$$

and the total inertia matrix $[I_O]$ can be computed as

$$[I_O] = \sum_{i=1}^3 [I_{Gi}]_{XYZ} + \sum_{i=1}^3 [I_{Ti}]_{XYZ} = \begin{bmatrix} 253.34 & 0 & 0 \\ 0 & 216.67 & 0 \\ 0 & 0 & 43.33 \end{bmatrix}_{XYZ} .$$

Moments of Inertia of Arbitrary Shape Objects

Two-Dimensional Solid Objects

Most two-dimensional objects can be approximated by assembling a set of triangular elements. The useful properties of a triangular element can be calculated by first defining the three vertices of the triangle as $\vec{r}_1 = \{x_k, y_k\}^T$, $\vec{r}_2 = \{x_l, y_l\}^T$ and $\vec{r}_3 = \{x_m, y_m\}^T$. The center of gravity of the element can then be computed as

$$\vec{R}_g = \frac{1}{3} (\vec{r}_1 + \vec{r}_2 + \vec{r}_3) \quad .$$

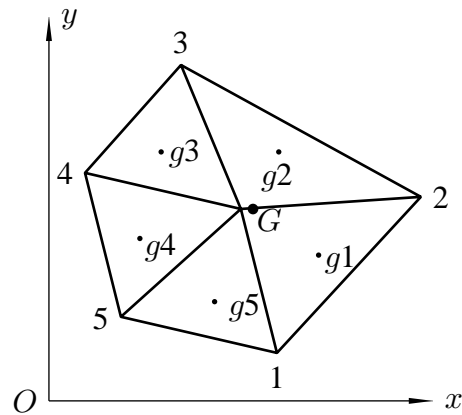
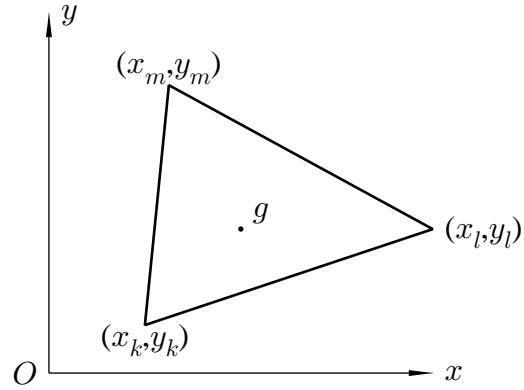
Its area, A , can be calculated as a determinant of a matrix as

$$A = \int \int dx dy = \frac{1}{2} \begin{vmatrix} 1 & x_k & y_k \\ 1 & x_l & y_l \\ 1 & x_m & y_m \end{vmatrix} .$$

If a constant area density, $\bar{\gamma}$, is used, the mass of the triangle is $m = \bar{\gamma}A$.

Define now the position vectors with respect to the center of mass as

$$\vec{\rho}_i = \vec{r}_i - \vec{R}_g \quad , \quad i = 1, 2, 3;$$



then the element's moment of inertia matrix about its center of gravity is

$$[I_g] = \frac{m}{12} \sum_{i=1}^3 (\vec{\rho}_i^T \vec{\rho}_i [1] - \vec{\rho}_i \vec{\rho}_i^T) = \frac{m}{12} \sum_{i=1}^3 \begin{bmatrix} \rho_{iy}^2 & -\rho_{ix}\rho_{iy} \\ -\rho_{iy}\rho_{ix} & \rho_{ix}^2 \end{bmatrix} .$$

Using the parallel axis theorem, the moment of inertia matrix, $[I_O]$, can be calculated as, $[I_O] = [I_G] + [I_T]$, in which the translation matrix, $[I_T]$, is defined as

$$[I_T] = m \left(\vec{R}_g^T \vec{R}_g [1] - \vec{R}_g \vec{R}_g^T \right) .$$

Assembling the Elements

The area of the object is first partitioned into M triangular subareas and the nodes are numbered and their x and y coordinates specified. The numbers used for the nodes are called global node numbers while the numbers of the three vertices, 1, 2 and 3, are called local node numbers. The way an element contributes to the entire object is established by three one-to-one pointers between the local and global nodes. For example, element 1 has nodes 1,2,6; element 2 has nodes 2,3,6; and element 5 has nodes 5,1,6; etc. Using these nodal coordinates, the moment of inertia matrix of the triangular elements, $[I_{gi}]$, can be calculated and translated to the origin,

O , using the position vectors, \vec{R}_{gi} . Therefore, the total moment of inertia and center of gravity for the object are,

$$[I_O] = \sum_{i=1}^M [I_{gi}] + [I_{Ti}] \quad \text{and} \quad \vec{R}_G = \left(\sum_{i=1}^M m_i \vec{R}_{gi} \right) / \left(\sum_{i=1}^M m_i \right) ,$$

respectively. Using the newly calculated global center of gravity, G , the moment of inertia matrix of the object can be determined at G by applying the parallel axis theorem as

$$[I_G] = [I_O] - [I_T] \quad ,$$

in which

$$[I_T] = \left(\sum_{i=1}^M m_i \right) \left(\vec{R}_G^T \vec{R}_G [1] - \vec{R}_G \vec{R}_G^T \right) .$$

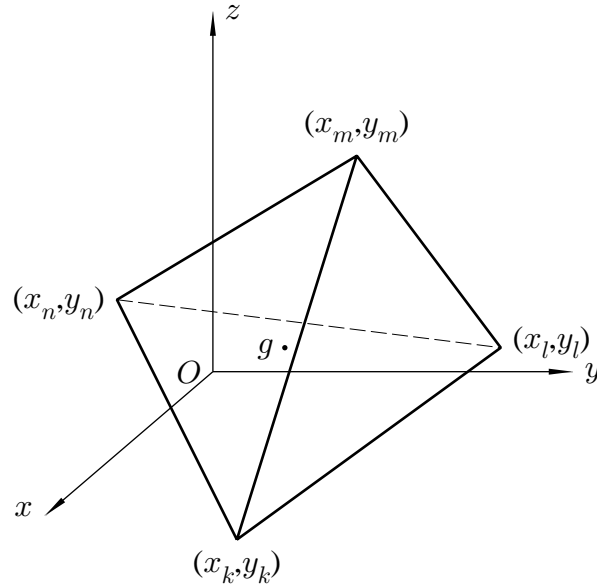
Three-Dimensional Solid Objects

Most three-dimensional objects can be approximated by assembling a set of tetrahedron elements. The useful properties of a tetrahedron element can be calculated by first defining the four vertices of the tetrahedron as $\vec{r}_1 = \{x_k, y_k\}^T$, $\vec{r}_2 = \{x_l, y_l\}^T$, $\vec{r}_3 = \{x_m, y_m\}^T$ and $\vec{r}_4 = \{x_n, y_n\}^T$. The center of gravity of the element can then be computed as

$$\vec{R}_g = \frac{1}{4} (\vec{r}_1 + \vec{r}_2 + \vec{r}_3 + \vec{r}_4) .$$

Its volume, V , can be calculated as a determinant of a matrix as

$$V = \int dV = \frac{1}{6} \begin{vmatrix} 1 & x_k & y_k & z_k \\ 1 & x_l & y_l & z_l \\ 1 & x_m & y_m & z_m \\ 1 & x_n & y_n & z_n \end{vmatrix} .$$



If a uniform mass density, $\bar{\gamma}$, is used, the mass of the tetrahedron is $m = \bar{\gamma}V$.

Define now the position vectors with respect to the center of mass as

$$\vec{\rho}_i = \vec{r}_i - \vec{R}_g \quad , \quad i = 1, 2, 3, 4;$$

then the element's moment of inertia matrix about its center of gravity is

$$[I_g] = \frac{m}{20} \sum_{i=1}^4 (\vec{\rho}_i^T \vec{\rho}_i [1] - \vec{\rho}_i \vec{\rho}_i^T) = \frac{m}{20} \sum_{i=1}^4 \begin{bmatrix} \rho_{iy}^2 + \rho_{iz}^2 & -\rho_{ix}\rho_{iy} & -\rho_{ix}\rho_{iz} \\ -\rho_{iy}\rho_{ix} & \rho_{ix}^2 + \rho_{iz}^2 & -\rho_{iy}\rho_{iz} \\ -\rho_{iz}\rho_{ix} & -\rho_{iz}\rho_{iy} & \rho_{ix}^2 + \rho_{iy}^2 \end{bmatrix} .$$

Using the parallel axis theorem, the moment of inertia matrix, $[I_O]$, can be calculated as, $[I_O] = [I_G] + [I_T]$, in which the translation matrix, $[I_T]$, is defined as

$$[I_T] = m \left(\vec{R}_g^T \vec{R}_g [1] - \vec{R}_g \vec{R}_g^T \right) .$$

```

write (*,*) 'Enter 2=2D, 3=3D'
read (*,*) ndim
if(ndim.eq.2)call twod
if(ndim.eq.3)call threed
end
-----
subroutine itgen(r,rg,n,it)
real r(n),rg(n),rho(3),it(n,n)
a=0.
do 1 i=1,n
rho(i)=r(i)-rg(i)
1 a=a+rho(i)**2
do 2 i=1,n
do 3 j=1,n
3 it(i,j)=-rho(i)*rho(j)
2 it(i,i)=a+it(i,i)
return
end
-----
subroutine clear(a,n)
real a(n)
do 1 i=1,n
1 a(i)=0.
return
end
-----
subroutine sum(a,b,n,c)
real a(n,n),b(n,n)
do 1 i=1,n
do 1 j=1,n
1 a(i,j)=a(i,j)+c*b(i,j)
return
end
-----
subroutine twod
parameter (mxnode=50)
real r(2,mxnode),ar(3,3)
real rgs(2),rgt(2),r0(2)
real itc(2,2),it(2,2),igs(2,2)
real igt(2,2),io(2,2)
integer ind(3)
call clear(r0,2)
call clear(io,4)
call clear(igt,4)
call clear(rgt,2)
areat=0.
open(1,file='input2d.dat',status='old')
read (1,*) nnode,nel
do 10 inode=1,nnode
10 read (1,*) (r(j,inode),j=1,2)
do 30 iel=1,nel
read (1,*) ind
call clear(igs,4)
do 21 j=1,2
21 rgs(j)=( r(j,ind(1))+r(j,ind(2))
# +r(j,ind(3)) )/3.
do 23 j=1,3
index=ind(j)
ar(j,1)=1.
do 22 i=1,2
22 ar(j,i+1)=r(i,index)
call itgen(r(1,index),rgs,2,itc)
call sum(igs,itc,2,1.)
23 continue
call det(ar,3,areas)
areas=abs(areas)/2.
call itgen(rgs,r0,2,it)
call sum(io,igs,2,areas/12.)
call sum(io,it,2,areas)
areat=areat+areas
do 24 j=1,2
24 rgt(j)=rgt(j)+areas*rgs(j)
30 continue
do 31 j=1,2
31 rgt(j)=rgt(j)/areat
call itgen(rgt,r0,2,it)
call sum(igt,io,2,1.)
call sum(igt,it,2,-areat)
write (*,41) rgt
41 format(' Centroid=',2f10.3//
# ' I(origin)',2lx,' I(centroid)')
do 50 i=1,2
# write (*,42) (io(i,j),j=1,2),
# (igt(i,j),j=1,2)
42 format(2f10.3,10x,2f10.3)
50 continue
return
end

```

```

-----
subroutine det(a,n,d)
real a(n,n)
d=1.
do 20 i=1,n-1
amax=abs(a(i,i))
imax=i
do 21 j=i+1,n
aji=abs(a(j,i))
if(aji.gt.amax)then
amax=aji
imax=j
endif
21 continue
if(imax.ne.i)then
do 22 k=i,n
temp=a(imax,k)
22 a(imax,k)=a(i,k)
a(i,k)=temp
d=-d
endif
do 20 j=i+1,n
alpha=a(j,i)/a(i,i)
do 10 k=i,n
10 a(j,k)=a(j,k)-alpha*a(i,k)
20 continue
do 30 i=1,n
30 d=d*a(i,i)
return
end
-----
subroutine threed
parameter (mxnode=50)
real r(3,mxnode),ar(4,4)
real rgs(3),rgt(3),r0(3)
real itc(3,3),it(3,3),igs(3,3)
real igt(3,3),io(3,3)
integer ind(4)
call clear(r0,3)
call clear(io,9)
call clear(igt,9)
call clear(rgt,3)
volumt=0.
open(1,file='input3d.dat',status='old')
read (1,*) nnode,nel
do 10 inode=1,nnode
10 read (1,*) (r(j,inode),j=1,3)
do 30 iel=1,nel
read (1,*) ind
call clear(igs,9)
do 21 j=1,3
21 rgs(j)=( r(j,ind(1))+r(j,ind(2))
# +r(j,ind(3))+r(j,ind(4)) )/4.
do 23 j=1,4
index=ind(j)
ar(j,1)=1.
do 22 i=1,3
22 ar(j,i+1)=r(i,index)
call itgen(r(1,index),rgs,3,itc)
call sum(igs,itc,3,1.)
23 continue
call det(ar,4,volums)
volums=abs(volums)/6.
call itgen(rgs,r0,3,it)
call sum(io,igs,3,volums/20.)
call sum(io,it,3,volums)
volumt=volumt+volums
do 24 j=1,3
24 rgt(j)=rgt(j)+volums*rgs(j)
30 continue
do 31 j=1,3
31 rgt(j)=rgt(j)/volumt
call itgen(rgt,r0,3,it)
call sum(igt,io,3,1.)
call sum(igt,it,3,-volumt)
write (*,41) rgt
41 format(' Centroid=',3f10.3//
# ' I(origin)',3lx,' I(centroid)')
do 50 i=1,3
# write (*,42) (io(i,j),j=1,3),
# (igt(i,j),j=1,3)
42 format(3f10.3,10x,3f10.3)
50 continue
return
end

```

Moments of Inertia of Thin Shells

Three-Dimensional Thin Shells

Most three-dimensional thin shells can be approximated by assembling a set of two-dimensional triangular shell elements. Unlike the two-dimensional triangular solid elements, the shell element vertices have x , y and z coordinates. The useful properties of a triangular element can be calculated by first defining the three vertices of the triangle as $\vec{r}_1 = \{x_k, y_k, z_k\}^T$, $\vec{r}_2 = \{x_l, y_l, z_l\}^T$ and $\vec{r}_3 = \{x_m, y_m, z_m\}^T$. The center of gravity of the element, in the xyz coordinate system, can then be computed as

$$\vec{R}_g = \frac{1}{3} (\vec{r}_1 + \vec{r}_2 + \vec{r}_3) \quad .$$

To calculate the element's area and its moments of inertia, it is necessary to define a local coordinate system, abc , so that the surface of the triangle lies on the ab -plane with the c -axis perpendicular to it. First define the unit vector, \hat{e}_a , as

$$\hat{e}_a = \frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|} \quad ,$$

then the unit vector, \hat{e}_c , as

$$\hat{e}_c = \frac{(\vec{r}_2 - \vec{r}_1) \times (\vec{r}_3 - \vec{r}_1)}{|(\vec{r}_2 - \vec{r}_1) \times (\vec{r}_3 - \vec{r}_1)|} \quad .$$

Because of the cross product, \hat{e}_c is perpendicular to \hat{e}_a , therefore, the unit vector, \hat{e}_b can be obtained as

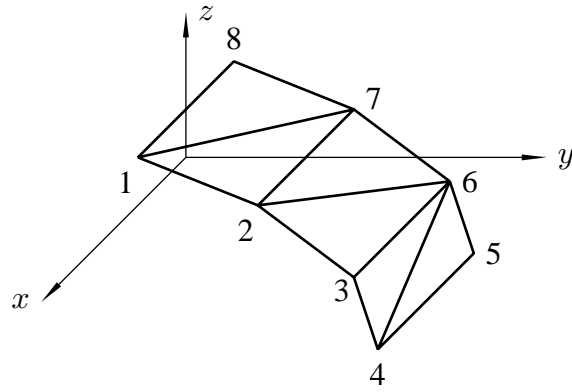
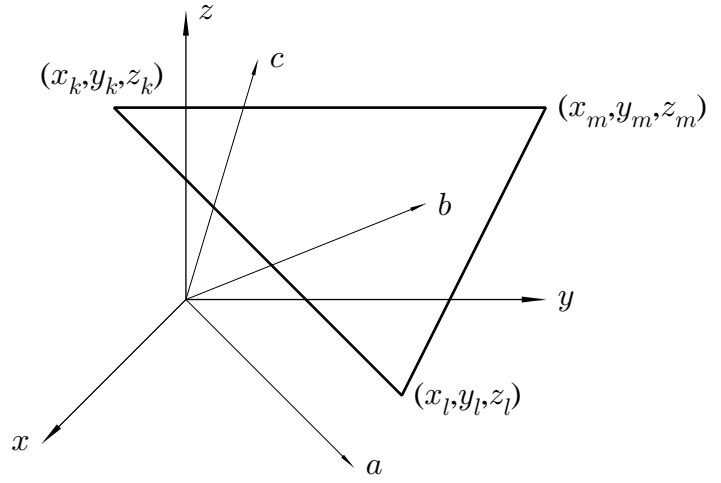
$$\hat{e}_b = \hat{e}_c \times \hat{e}_a \quad .$$

Using these three unit vectors, a transformation matrix can be developed between the two coordinate systems as follows:

$$[Q(abc \leftarrow xyz)] = \begin{bmatrix} \hat{e}_a^T \\ \hat{e}_b^T \\ \hat{e}_c^T \end{bmatrix} \quad .$$

Now transform the coordinates of the vertices from the xyz system to the abc system using the matrix products:

$$\begin{Bmatrix} a_k \\ b_k \\ c_k \end{Bmatrix} = [Q] \begin{Bmatrix} x_k \\ y_k \\ z_k \end{Bmatrix} \quad , \quad \begin{Bmatrix} a_l \\ b_l \\ c_l \end{Bmatrix} = [Q] \begin{Bmatrix} x_l \\ y_l \\ z_l \end{Bmatrix} \quad , \quad \begin{Bmatrix} a_m \\ b_m \\ c_m \end{Bmatrix} = [Q] \begin{Bmatrix} x_m \\ y_m \\ z_m \end{Bmatrix} \quad .$$



Because \hat{e}_c is perpendicular to the surface, the ordinates, c_k , c_l and c_m should be the same. The element's area, A , can then be calculated as a determinant of a matrix as

$$A = \int \int da db = \frac{1}{2} \begin{vmatrix} 1 & a_k & b_k \\ 1 & a_l & b_l \\ 1 & a_m & b_m \end{vmatrix} .$$

If a constant area density, $\bar{\gamma}$, is used, the mass of the triangle is $m = \bar{\gamma}A$.

Define now the position vectors with respect to the center of mass in the abc system as

$$\vec{\rho}_i = [Q](\vec{r}_i - \vec{R}_g) \quad , \quad i = 1, 2, 3;$$

then the element's moment of inertia matrix about its center of gravity is

$$[I_g]_{abc} = \frac{m}{12} \sum_{i=1}^3 (\vec{\rho}_i^T \vec{\rho}_i [1] - \vec{\rho}_i \vec{\rho}_i^T) = \frac{m}{12} \sum_{i=1}^3 \begin{bmatrix} \rho_{ib}^2 & -\rho_{ia}\rho_{ib} \\ -\rho_{ib}\rho_{ia} & \rho_{ia}^2 \end{bmatrix} .$$

After the transformation,

$$[I_g]_{xyz} = [Q]^T [I_g]_{abc} [Q] \quad ,$$

the moment of inertia matrix, $[I_O]$, can be calculated as, $[I_O] = [I_g]_{xyz} + [I_T]$, using the parallel axis theorem. The translation matrix, $[I_T]$, is defined as

$$[I_T] = m \left(\vec{R}_g^T \vec{R}_g [1] - \vec{R}_g \vec{R}_g^T \right) .$$

Assembling the Elements

The surface area of the thin shell is first partitioned into M triangular subareas and the nodes are numbered and their x , y and z coordinates specified. The numbers used for the nodes are called global node numbers while the numbers of the three vertices, 1, 2 and 3, are called local node numbers. The way an element contributes to the entire object is established by three one-to-one pointers between the local and global nodes. For example, element 1 has nodes 1,7,8; element 2 has nodes 1,2,7; and element 5 has nodes 3,4,6; etc. Using these nodal coordinates in the xyz system, a "custom made" abc coordinate system can be fitted to a particular triangular element so that the moment of inertia matrix, $[I_{gi}]$, can first be calculated in the abc system and then transformed to the xyz system. After $[I_{gi}]$ is obtained in the xyz system, it can be translated to the origin, O , using the parallel axis theorem and the position vector, \vec{R}_{gi} . Therefore, the total moment of inertia and center of gravity for the object are,

$$[I_O] = \sum_{i=1}^M [I_{gi}] + [I_{Ti}] \quad \text{and} \quad \vec{R}_G = \left(\sum_{i=1}^M m_i \vec{R}_{gi} \right) / \left(\sum_{i=1}^M m_i \right) ,$$

respectively. Using the newly calculated global center of gravity, G , the moment of inertia matrix of the object can be determined at G by applying the parallel axis theorem as

$$[I_G] = [I_O] - [I_T] \quad ,$$

in which

$$[I_T] = \left(\sum_{i=1}^M m_i \right) \left(\vec{R}_G^T \vec{R}_G [1] - \vec{R}_G \vec{R}_G^T \right) .$$

```

parameter (mxnode=100)
common/newvec/qr(3,3),q(3,3),qt(3,3)
real r(3,mxnode),ar(3,3)
real qrgs(3),rgs(3),rgt(3),r0(3)
real itc(3,3),it(3,3),igs(3,3)
real igt(3,3),io(3,3)
integer ind(3)
call clear(r0,3)
call clear(io,9)
call clear(igt,9)
call clear(rgt,3)
areat=0.
open(1,file='shell.dat',status='old')
read(1,*) nnode,nel
do 10 inode=1,nnode
10 read(1,*) (r(j,inode),j=1,3)
do 30 iel=1,nel
read(1,*) ind
call clear(igs,9)
call newcor(r(1,ind(1)),r(1,ind(2)),r(1,ind(3)))
do 21 j=1,3
21 rgs(j)=(r(j,ind(1))+r(j,ind(2))+r(j,ind(3)))/3.
qrgs(j)=(qr(j,1)+qr(j,2)+qr(j,3))/3.
do 23 j=1,3
ar(j,1)=1.
do 22 i=1,2
22 ar(j,i+1)=qr(i,j)
call itgen(qr(1,j),qrgs,3,itc)
itc(3,3)=0.0
call sum(igs,itc,3,1.)
23 continue
call det(ar,3,areas)
areas=abs(areas)/2.
call itgen(rgs,r0,3,it)
call rotate(igs)
call sum(io,igs,3,areas/12.)
call sum(io,it,3,areas)
areat=areat+areas
do 24 j=1,3
24 rgt(j)=rgt(j)+areas*rgs(j)
30 continue
do 31 j=1,3
31 rgt(j)=rgt(j)/areat
call itgen(rgt,r0,3,it)
call sum(igt,io,3,1.)
call sum(igt,it,3,-areat)
write(*,41) rgt
41 format(' Centroid=' ,3f10.3//
# ' I(origin)',3lx,' I(centroid)')
do 50 i=1,3
write(*,42) (io(i,j),j=1,3),
# (igt(i,j),j=1,3)
42 format(3f10.3,10x,3f10.3)
50 continue
stop
end

c-----
subroutine newcor(r1,r2,r3)
common/newvec/qr(3,3),q(3,3),qt(3,3)
real r1(3),r2(3),r3(3)
real d1(3),d2(3),d3(3)
real qrg(3)
do 10 i=1,3
10 d1(i)=r2(i)-r1(i)
d2(i)=r3(i)-r1(i)
continue
call cross(d1,d2,d3)
call norm(d1)
call norm(d3)
call cross(d3,d1,d2)
do 20 i=1,3
20 q(1,i)=d1(i)
q(2,i)=d2(i)
q(3,i)=d3(i)
qt(i,1)=d1(i)
qt(i,2)=d2(i)
qt(i,3)=d3(i)
continue
call trans(q,r1,qr(1,1))
call trans(q,r2,qr(1,2))
call trans(q,r3,qr(1,3))
return
end

c-----
subroutine cross(a,b,c)
real a(3),b(3),c(3)
c(1)=a(2)*b(3)-a(3)*b(2)
c(2)=a(3)*b(1)-a(1)*b(3)
c(3)=a(1)*b(2)-a(2)*b(1)
return
end

c-----
subroutine norm(a)
real a(3)
al=0.
do 10 i=1,3
10 al=al+a(i)**2
al=sqrt(al)
do 20 i=1,3
20 a(i)=a(i)/al
return
end

c-----
subroutine trans(q,a,b)
real q(3,3),a(3),b(3)
do 10 i=1,3
b(i)=0.0
do 10 j=1,3
10 b(i)=b(i)+q(i,j)*a(j)
return
end

c-----
subroutine rotate(a)
common/newvec/qr(3,3),q(3,3),qt(3,3)
real a(3,3),temp(3,3)
do 10 i=1,3
do 10 j=1,3
temp(i,j)=0.0
do 10 k=1,3
10 temp(i,j)=temp(i,j)+a(i,k)*q(k,j)
do 20 i=1,3
do 20 j=1,3
a(i,j)=0.0
do 20 k=1,3
20 a(i,j)=a(i,j)+qt(i,k)*temp(k,j)
return
end

c-----
subroutine itgen(r,rg,n,it)
real r(n),rg(n),rho(3),it(n,n)
a=0.
do 1 i=1,n
rho(i)=r(i)-rg(i)
a=a+rho(i)**2
do 2 i=1,n
do 3 j=1,n
3 it(i,j)=-rho(i)*rho(j)
2 it(i,i)=a+it(i,i)
return
end

c-----
subroutine clear(a,n)
real a(n)
do 1 i=1,n
1 a(i)=0.
return
end

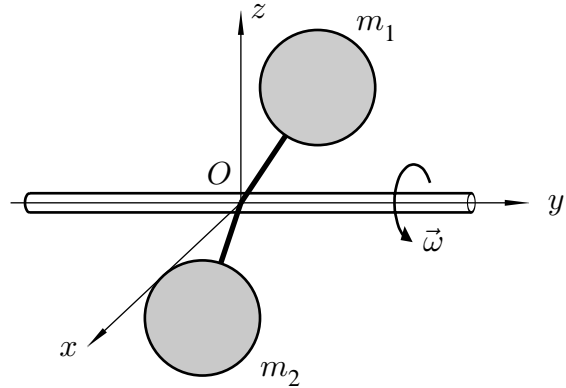
c-----
subroutine sum(a,b,n,c)
real a(n,n),b(n,n)
do 1 i=1,n
do 1 j=1,n
1 a(i,j)=a(i,j)+c*b(i,j)
return
end

c-----
subroutine det(a,n,d)
real a(n,n)
d=1.
do 20 i=1,n-1
amax=abs(a(i,i))
imax=i
do 21 j=i+1,n
aji=abs(a(j,i))
if(aji.gt.amax)then
amax=aji
imax=j
endif
21 continue
if(imax.ne.i)then
do 22 k=i,n
temp=a(imax,k)
a(imax,k)=a(i,k)
a(i,k)=temp
d=-d
endif
do 20 j=i+1,n
alpha=a(j,i)/a(i,i)
do 10 k=i,n
10 a(j,k)=a(j,k)-alpha*a(i,k)
20 continue
do 30 i=1,n
30 d=d*a(i,i)
return
end

```

Example – Off-Balanced Rotating Object

Two spheres of mass 5 kg and radius 1.5 m are mounted on a massless rod as shown in the figure. The position vectors of m_1 and m_2 are respectively, $\vec{r}_1 = \{0, 2, 3\}^T$ and $\vec{r}_2 = \{0, -1, -3\}^T$. Find



- the moment of inertia matrix about point O using the xyz system, and
- the moment vector that the spheres exert on the rod at point O if the rod is rotating with an angular velocity of ω .

Solution:

Part (a)

$[I_G]$ of both spheres 1 and 2 are

$$[I_G]_1 = [I_G]_2 = \frac{2}{5}mR^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \frac{2}{5}(5)(1.5)^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4.5 & 0 & 0 \\ 0 & 4.5 & 0 \\ 0 & 0 & 4.5 \end{bmatrix}.$$

Now calculate $[I_T]$ (parallel axis theorem) for both spheres to move origin to point O .

$[I_T]$ for sphere 1 using $\vec{R}_{G1} = \vec{r}_1 = \{0, 2, 3\}^T$ is

$$[I_T]_1 = 5 \left((0^2 + 2^2 + 3^2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 6 \\ 0 & 6 & 9 \end{bmatrix} \right) = 5 \begin{bmatrix} 13 & 0 & 0 \\ 0 & 9 & -6 \\ 0 & -6 & 4 \end{bmatrix}$$

$[I_T]$ for sphere 2 using $\vec{R}_{G2} = \vec{r}_2 = \{0, -1, -3\}^T$ is

$$[I_T]_2 = 5 \left((0^2 + (-1)^2 + (-3)^2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & 9 \end{bmatrix} \right) = 5 \begin{bmatrix} 10 & 0 & 0 \\ 0 & 9 & -3 \\ 0 & -3 & 1 \end{bmatrix}$$

The moment of inertia matrix $[I_O]$ using the xyz system is then

$$\begin{aligned} [I_O] &= ([I_G]_1 + [I_G]_2) + ([I_T]_1 + [I_T]_2) \\ &= \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} + 5 \begin{bmatrix} 23 & 0 & 0 \\ 0 & 18 & -9 \\ 0 & -9 & 5 \end{bmatrix} = \begin{bmatrix} 124 & 0 & 0 \\ 0 & 99 & -45 \\ 0 & -45 & 34 \end{bmatrix} \end{aligned}$$

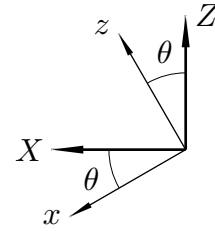
Part (b)

As the rod rotates with angular velocity $\omega \hat{j}$, it is convenient to attach system xyz so that it rotates with the masses, i.e., let the angular velocity of the coordinate system be $\vec{\Omega} = \omega \hat{j} = \omega \hat{J}$.

Note: In this case $\vec{\Omega}$ is the same as the angular velocity of the body. For some other problems, however, the body may rotate differently from the coordinate system, so a distinction must be made between $\vec{\Omega}$ and $\vec{\omega}$.

With the xyz system rotating with respect to system XYZ , the transformation matrix can be defined as

$$\begin{Bmatrix} q_X \\ q_Y \\ q_Z \end{Bmatrix} \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix} \begin{Bmatrix} q_x \\ q_y \\ q_z \end{Bmatrix} = [Q] \begin{Bmatrix} q_x \\ q_y \\ q_z \end{Bmatrix}$$



in which $\theta = \omega t$.

Method 1 Use moment of inertia matrix $[I_O]_{XYZ}$:

$$\begin{aligned} [I_O]_{XYZ} &= [Q][I_O]_{xyz}[Q]^T = [Q] \begin{bmatrix} 124 & 0 & 0 \\ 0 & 99 & -45 \\ 0 & -45 & 34 \end{bmatrix} \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix} \\ &= \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix} \begin{bmatrix} 124 \cos \theta & 0 & -124 \sin \theta \\ -45 \sin \theta & 99 & -45 \cos \theta \\ 34 \sin \theta & -45 & 34 \cos \theta \end{bmatrix} \\ &= \begin{bmatrix} 124 \cos^2 \theta + 34 \sin^2 \theta & -45 \sin \theta & -90 \sin \theta \cos \theta \\ -45 \sin \theta & 99 & -45 \cos \theta \\ -90 \sin \theta \cos \theta & -45 \cos \theta & 124 \sin^2 \theta + 34 \cos^2 \theta \end{bmatrix} \end{aligned}$$

A moment of inertia matrix which is a function of time t as $\theta = \omega t$. Now, the angular momentum vector can be calculated as

$$\begin{Bmatrix} H_{OX} \\ H_{OY} \\ H_{OZ} \end{Bmatrix} = [I_O]_{XYZ} \begin{Bmatrix} \omega_X \\ \omega_Y \\ \omega_Z \end{Bmatrix} = [I_O]_{XYZ} \begin{Bmatrix} 0 \\ \omega \\ 0 \end{Bmatrix}_{XYZ} = \begin{Bmatrix} -45\omega \sin \omega t \\ 99\omega \\ -45\omega \cos \omega t \end{Bmatrix}_{XYZ}$$

and the moment vector $\sum \vec{M}_O$ about point O is simply

$$\sum \begin{Bmatrix} M_{OX} \\ M_{OY} \\ M_{OZ} \end{Bmatrix} = \frac{d}{dt} \begin{Bmatrix} H_{OX} \\ H_{OY} \\ H_{OZ} \end{Bmatrix} = \begin{Bmatrix} -45\omega^2 \cos \omega t \\ 0 \\ 45\omega^2 \sin \omega t \end{Bmatrix}_{XYZ}$$

Method 2 Use moment of inertia matrix $[I_O]_{xyz}$. Calculate the angular momentum vector in the xyz system:

$$\begin{Bmatrix} H_{Ox} \\ H_{Oy} \\ H_{Oz} \end{Bmatrix} = [I_O]_{xyz} \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} = \begin{bmatrix} 124 & 0 & 0 \\ 0 & 99 & -45 \\ 0 & -45 & 34 \end{bmatrix} \begin{Bmatrix} 0 \\ \omega \\ 0 \end{Bmatrix}_{xyz} = \begin{Bmatrix} 0 \\ 99\omega \\ -45\omega \end{Bmatrix}_{xyz}$$

To calculate the moment vector, a time derivative must be taken with respect to a rotating basis:

$$\begin{Bmatrix} M_{Ox} \\ M_{Oy} \\ M_{Oz} \end{Bmatrix} = \frac{d_c}{dt} \begin{Bmatrix} H_{Ox} \\ H_{Oy} \\ H_{Oz} \end{Bmatrix} + \vec{\Omega} \times \begin{Bmatrix} H_{Ox} \\ H_{Oy} \\ H_{Oz} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ \omega \\ 0 \end{Bmatrix} \times \begin{Bmatrix} 0 \\ 99\omega \\ -45\omega \end{Bmatrix} = \begin{Bmatrix} -45\omega^2 \\ 0 \\ 0 \end{Bmatrix}_{xyz}$$

and the same moment vector in the XYZ system is

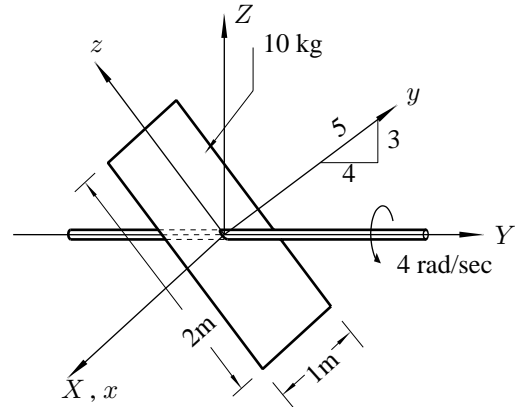
$$\sum \begin{Bmatrix} M_{OX} \\ M_{OY} \\ M_{OZ} \end{Bmatrix} = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix} \begin{Bmatrix} -45\omega^2 \\ 0 \\ 0 \end{Bmatrix}_{xyz} = \begin{Bmatrix} -45\omega^2 \cos \omega t \\ 0 \\ 45\omega^2 \sin \omega t \end{Bmatrix}_{XYZ}$$

Example – Badly Aligned Rectangular Plate

As the rod is rotating about the Y -axis with an angular velocity of $\vec{\Omega} = 4\hat{J}$ rad/sec, the angle θ between the x and the X axes is $\theta = 4t$. The xyz coordinate system is convenient for the calculation of the moment of inertia matrix, i.e.,

$$\begin{aligned} [I_O]_{xyz} &= \frac{1}{12}(10) \begin{bmatrix} 2^2 & 0 & 0 \\ 0 & 2^2 + 1^2 & 0 \\ 0 & 0 & 1^2 \end{bmatrix} \\ &= \begin{bmatrix} 3.33 & 0 & 0 \\ 0 & 4.17 & 0 \\ 0 & 0 & 0.83 \end{bmatrix} \end{aligned}$$

while the XYZ system is needed to determine the absolute bending moment.



The transformation matrix $[Q(xyz \leftarrow XYZ)]$ can be determined as a product between two simpler transformation as:

$$[Q] = \begin{bmatrix} \cos 4t & 0 & \sin 4t \\ 0 & 1 & 0 \\ -\sin 4t & 0 & \cos 4t \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.8 & -0.6 \\ 0 & 0.6 & 0.8 \end{bmatrix} = \begin{bmatrix} \cos 4t & 0.6 \sin 4t & 0.8 \sin 4t \\ 0 & 0.8 & -0.6 \\ -\sin 4t & 0.6 \cos 4t & 0.8 \cos 4t \end{bmatrix}$$

Method 1 – Use fixed coordinate system XYZ .

First we must evaluate the moment of inertia matrix $[I_O]_{XYZ}$ as a function of time. This matrix fluctuates with time because the geometry is constantly changing.

$$[I_O]_{XYZ} = [Q][I_O]_{xyz}[Q]^T$$

or

$$\begin{aligned} [I_O]_{XYZ} &= [Q] \begin{bmatrix} 3.33 & 0 & 0 \\ 0 & 4.17 & 0 \\ 0 & 0 & 0.83 \end{bmatrix} \begin{bmatrix} \cos 4t & 0 & -\sin 4t \\ 0.6 \sin 4t & 0.8 & 0.6 \cos 4t \\ 0.8 \sin 4t & -0.6 & 0.8 \cos 4t \end{bmatrix} \\ &= \begin{bmatrix} \cos 4t & 0.6 \sin 4t & 0.8 \sin 4t \\ 0 & 0.8 & -0.6 \\ -\sin 4t & 0.6 \cos 4t & 0.8 \cos 4t \end{bmatrix} \begin{bmatrix} 3.33 \cos 4t & 0 & -3.33 \sin 4t \\ 2.50 \sin 4t & 3.34 & 2.50 \cos 4t \\ 0.67 \sin 4t & -0.50 & 0.67 \cos 4t \end{bmatrix} \\ &= \begin{bmatrix} 3.33 \cos^2 4t + 2 \sin^2 4t & 1.6 \sin 4t & -1.29 \sin 4t \cos 4t \\ 1.6 \sin 4t & 3 & 1.6 \cos 4t \\ -1.29 \sin 4t \cos 4t & 1.6 \cos 4t & 3.33 \sin^2 4t + 2 \cos^2 4t \end{bmatrix} \end{aligned}$$

Subsequently, the angular momentum in the XYZ system can readily be calculated as

$$\{H_O\}_{XYZ} = [I_O]_{XYZ}\{\omega\}_{XYZ} = \begin{bmatrix} - & - & - & 1.6 \sin 4t & - & - & - \\ - & - & - & 3 & - & - & - \\ - & - & - & 1.6 \cos 4t & - & - & - \end{bmatrix} \begin{Bmatrix} 0 \\ 4 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 6.4 \sin 4t \\ 12 \\ 6.4 \cos 4t \end{Bmatrix}_{XYZ}$$

The moment vector about point O can now be obtained by differentiating the angular momentum vector as follows

$$\left\{ \sum M_O \right\}_{XYZ} = \frac{d}{dt} \{H_O\}_{XYZ} = \begin{Bmatrix} 25.6 \cos 4t \\ 0 \\ -25.6 \sin 4t \end{Bmatrix}_{XYZ}$$

It is clear that oscillating moments about the X and Z axes are caused by the mis-aligned plate trying to “right” itself. There is no torque in the Y direction because the rod is rotating at a constant angular velocity and there is no friction considered.

Method 2 – Use rotating coordinate system xyz .

In the xyz system,

$$\{H_O\}_{xyz} = [I_O]_{xyz}\{\omega\}_{xyz}$$

in which

$$\{\omega\}_{xyz} = [Q]^T \begin{Bmatrix} 0 \\ 4 \\ 0 \end{Bmatrix}_{XYZ} = \begin{bmatrix} \cos 4t & 0 & -\sin 4t \\ 0.6 \sin 4t & 0.8 & 0.6 \cos 4t \\ 0.8 \sin 4t & -0.6 & 0.8 \cos 4t \end{bmatrix} \begin{Bmatrix} 0 \\ 4 \\ 0 \end{Bmatrix}_{XYZ} = \begin{Bmatrix} 0 \\ 3.2 \\ -2.4 \end{Bmatrix}_{xyz}$$

Therefore

$$\{H_O\}_{xyz} = \begin{bmatrix} 3.33 & 0 & 0 \\ 0 & 4.17 & 0 \\ 0 & 0 & 0.83 \end{bmatrix} \begin{Bmatrix} 0 \\ 3.2 \\ -2.4 \end{Bmatrix}_{xyz} = \begin{Bmatrix} 0 \\ 13.33 \\ -2.0 \end{Bmatrix}_{xyz}$$

Now to differentiate \vec{H}_O with respect to a rotating coordinate system, use the formulation

$$\{M_O\}_{xyz} = \frac{d_c}{dt} \begin{Bmatrix} 0 \\ 13.33 \\ -2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 3.2 \\ -2.4 \end{Bmatrix} \times \begin{Bmatrix} 0 \\ 13.33 \\ -2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 25.6 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 25.6 \\ 0 \\ 0 \end{Bmatrix}_{xyz}$$

The “righting” moment is about the rotating x axis only. To obtain the results in the XYZ system, use the transformation:

$$\left\{ \sum M_O \right\} = [Q] \begin{Bmatrix} 25.6 \\ 0 \\ 0 \end{Bmatrix}_{xyz} = \begin{bmatrix} \cos 4t & 0.6 \sin 4t & 0.8 \sin 4t \\ 0 & 0.8 & -0.6 \\ -\sin 4t & 0.6 \cos 4t & 0.8 \cos 4t \end{bmatrix} \begin{Bmatrix} 25.6 \\ 0 \\ 0 \end{Bmatrix}_{xyz}$$

or

$$\left\{ \sum M_O \right\} = \begin{Bmatrix} 25.6 \cos 4t \\ 0 \\ -25.6 \sin 4t \end{Bmatrix}_{XYZ}$$

Kinetic Energy of a Rigid Body

Since a rigid body is a special case of a system of particles, the total kinetic energy of a rigid body can be expressed generally as

$$T = \frac{1}{2} \sum_{i=1}^N m_i \vec{v}_i^2 \quad ,$$

but the additional restrictions of a rigid body requires that

$$\vec{v}_i = \vec{v}_G + \vec{\omega} \times \vec{\rho}_i \quad ,$$

therefore,

$$\begin{aligned} T &= \frac{1}{2} \sum_{i=1}^N m_i (\vec{v}_G + \vec{\omega} \times \vec{\rho}_i) \cdot (\vec{v}_G + \vec{\omega} \times \vec{\rho}_i) \\ &= \frac{1}{2} \left(\sum_{i=1}^N m_i \right) v_G^2 + \sum_{i=1}^N m_i \vec{v}_G \cdot (\vec{\omega} \times \vec{\rho}_i) + \frac{1}{2} \sum_{i=1}^N m_i (\vec{\omega} \times \vec{\rho}_i) \cdot (\vec{\omega} \times \vec{\rho}_i) \end{aligned}$$

The second term of the expression can be eliminated by the definition of the center of mass because

$$\sum_{i=1}^N m_i \vec{v}_G \cdot (\vec{\omega} \times \vec{\rho}_i) = \vec{v}_G \cdot \vec{\omega} \times \sum_{i=1}^N m_i \vec{\rho}_i = \vec{0} \quad .$$

Introduce now the vector identity for four vectors:

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = \vec{c} \cdot [\vec{d} \times (\vec{a} \times \vec{b})] \quad ;$$

apply it to the third term of the expression by assigning: $\vec{a} = \vec{\omega}$, $\vec{b} = \vec{\rho}_i$, $\vec{c} = \vec{\omega}$ and $\vec{d} = \vec{\rho}_i$ to yield

$$(\vec{\omega} \times \vec{\rho}_i) \cdot (\vec{\omega} \times \vec{\rho}_i) = \vec{\omega} \cdot (\vec{\rho}_i \times \vec{\omega} \times \vec{\rho}_i) \quad .$$

Therefore, the kinetic energy of a rigid body can be represented as

$$T = \frac{1}{2} M v_G^2 + \frac{1}{2} \vec{\omega} \cdot \sum_{i=1}^N m_i (\vec{\rho}_i \times \vec{\omega} \times \vec{\rho}_i) \quad ,$$

or, by the previous definition of a moment of inertia matrix,

$$T = \frac{1}{2} M v_G^2 + \frac{1}{2} \vec{\omega} \cdot [I_G] \vec{\omega} \quad ,$$

or

$$T = \frac{1}{2} \vec{v}_G \cdot \vec{G}_G + \frac{1}{2} \vec{\omega} \cdot \vec{H}_G \quad ,$$

in terms of the linear and angular momenta.

Special Case: Two-Dimensional Rigid Body Problems

The two-dimensional rigid body problem is just a special case of the three-dimensional problems, but since most textbook problems are two-dimensional, it is worthwhile to discuss the simplifications that it offers.

In three dimensions, the equations of motion, three degrees of freedom each, are

$$\sum \vec{F} = m\vec{a}_G \quad \text{and} \quad \sum \vec{M}_O = [I_G]\vec{\alpha} + \vec{R}_G \times m\vec{a}_G \quad .$$

In two dimensions, the vectors are simplified as follows:

$$\vec{F} = \begin{Bmatrix} F_x \\ F_y \\ 0 \end{Bmatrix} \quad , \quad \vec{a}_G = \begin{Bmatrix} a_{Gx} \\ a_{Gy} \\ 0 \end{Bmatrix} \quad , \quad \vec{R}_G = \begin{Bmatrix} r_{Gx} \\ r_{Gy} \\ 0 \end{Bmatrix} \quad \text{and} \quad \vec{\alpha} = \begin{Bmatrix} 0 \\ 0 \\ \alpha_z \end{Bmatrix} \quad ,$$

and the moment of inertia matrix is simplified to

$$[I_G] = \begin{bmatrix} I_{Gxx} & I_{Gxy} & 0 \\ I_{Gyx} & I_{Gyy} & 0 \\ 0 & 0 & I_{Gzz} \end{bmatrix} \quad .$$

Therefore, the translation equation of motion simplifies to

$$\sum \begin{Bmatrix} F_x \\ F_y \end{Bmatrix} = m \begin{Bmatrix} a_{Gx} \\ a_{Gy} \end{Bmatrix}$$

and the rotation equation of motion reduces to

$$\begin{aligned} \sum \begin{Bmatrix} M_{Ox} \\ M_{Oy} \\ M_{Oz} \end{Bmatrix} &= \begin{bmatrix} I_{Gxx} & I_{Gxy} & 0 \\ I_{Gyx} & I_{Gyy} & 0 \\ 0 & 0 & I_{Gzz} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \alpha_z \end{Bmatrix} + \begin{Bmatrix} r_{Gx} \\ r_{Gy} \\ 0 \end{Bmatrix} \times m \begin{Bmatrix} a_{Gx} \\ a_{Gy} \\ 0 \end{Bmatrix} \\ &= \begin{Bmatrix} 0 \\ 0 \\ I_{Gzz}\alpha_z \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ m(r_{Gx}a_{Gy} - r_{Gy}a_{Gx}) \end{Bmatrix} \quad . \end{aligned}$$

The lack of angular momenta in the x and the y directions require that $M_{Ox} = 0$ and $M_{Oy} = 0$. Thus, the rotation equation of motion becomes a scalar one, i.e.,

$$\sum M_{Oz} = I_{Gzz}\alpha_z + m(r_{Gx}a_{Gy} - r_{Gy}a_{Gx}) \quad .$$

For the special situation when point O is fixed, the parallel axis theorem can be employed by calculating the translation matrix:

$$[I_T] = m \begin{bmatrix} r_{Gy}^2 & -r_{Gx}r_{Gy} & 0 \\ -r_{Gy}r_{Gx} & r_{Gx}^2 & 0 \\ 0 & 0 & r_{Gx}^2 + r_{Gy}^2 \end{bmatrix} \quad ,$$

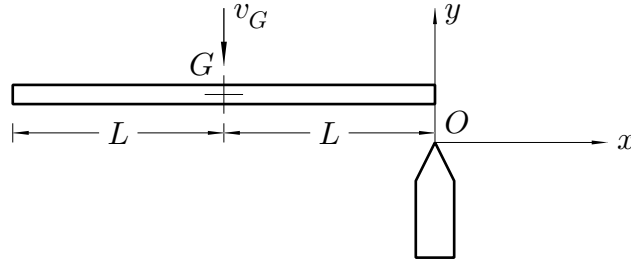
but since $\vec{\alpha}$ only has a z -component, the only term needed for the moment of inertia matrix is

$$I_{Ozz} = I_{Gzz} + m(r_{Gx}^2 + r_{Gy}^2) \quad .$$

The above expression is the parallel-axis theorem that is the most familiar for beginning classical physics students.

Example – Conservation of Angular Momentum I

A slender rod of length $2L$ is dropping with a velocity of \vec{v}_G when its right end hits a fixed pole. Find the angular velocity $\vec{\omega}_2$ immediately after the collision if the collision is plastic in nature.



Solution:

Use the conservation of angular momentum about point O because the force of the collision has a zero moment there. With the summation of external moments equal to zero, the equation for angular impulse and angular momentum implies

$$\vec{H}_{O1} = \vec{H}_{O2}$$

Before Collision:

$$\vec{H}_{O1} = \vec{R}_G \times m\vec{v}_{G1} + [I_G]\vec{\omega}_1$$

in which $\vec{\omega}_1 = \vec{0}$ initially. Substitute the position vector $\vec{R}_G = \{-L, 0, 0\}^T$ in the above equation, we have

$$\vec{H}_{O1} = \begin{Bmatrix} -L \\ 0 \\ 0 \end{Bmatrix} \times m \begin{Bmatrix} 0 \\ -v_G \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ mLv_G \end{Bmatrix}$$

After Collision:

$$\vec{H}_{O2} = \vec{R}_G \times m\vec{v}_{G2} + [I_G]\vec{\omega}_2$$

in which $\vec{\omega}_2$ and \vec{v}_{G2} are both unknowns. Since motion of all points on a rigid body can be determined with just one rotation vector and one translation vector, we shall use the known factor that $\vec{v}_O = \vec{0}$ because it is a fixed point at the time of impact. We can now write

$$\vec{v}_{G2} = \vec{0} + \vec{\omega}_2 \times \vec{R}_{G/O} = \vec{\omega}_2 \times \vec{R}_G$$

or

$$\vec{H}_{O2} = \vec{R}_G \times m(\vec{\omega}_2 \times \vec{R}_G) + [I_G]\vec{\omega}_2 = [I_O]\vec{\omega}_2$$

in which the moment of inertia matrix about the fixed point O can be determined as

$$[I_O] = \frac{1}{12}m \begin{bmatrix} 0 & 0 & 0 \\ 0 & (2L)^2 & 0 \\ 0 & 0 & (2L)^2 \end{bmatrix} + m \left(L^2[1] - \begin{bmatrix} L^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{4}{3}mL^2 & 0 \\ 0 & 0 & \frac{4}{3}mL^2 \end{bmatrix}.$$

It follows that

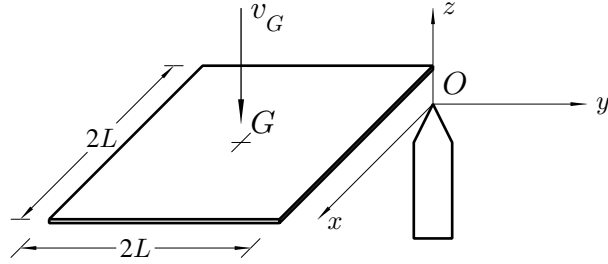
$$\vec{H}_{O1} = \begin{Bmatrix} 0 \\ 0 \\ mLv_G \end{Bmatrix} = \vec{H}_{O2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{4}{3}mL^2 & 0 \\ 0 & 0 & \frac{4}{3}mL^2 \end{bmatrix} \begin{Bmatrix} \omega_{x2} \\ \omega_{y2} \\ \omega_{z2} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \frac{4}{3}mL^2\omega_{y2} \\ \frac{4}{3}mL^2\omega_{z2} \end{Bmatrix}$$

which implies $\omega_{y2} = 0$ and $\omega_{z2} = mLv_G/(4mL^2/3)$, or

$$\vec{\omega}_2 = \left\{ 0, 0, \frac{3v_G}{4L} \right\}^T$$

Example – Conservation of Angular Momentum II

A thin square plate of dimensions $2L \times 2L$ is dropping with a velocity of \vec{v}_G when one of its corners collides with a fixed pole. Find the angular velocity $\vec{\omega}_2$ immediately after the collision if the collision is plastic in nature.



Solution:

Use the conservation of angular momentum about point O because the force of the collision has a zero moment there. With the summation of external moments equal to zero, the equation for angular impulse and angular momentum implies

$$\vec{H}_{O1} = \vec{H}_{O2}$$

Before Collision:

$$\vec{H}_{O1} = \vec{R}_G \times m\vec{v}_{G1} + [I_G]\vec{\omega}_1$$

in which $\vec{\omega}_1 = \vec{0}$ initially. Use the position vector $\vec{R}_G = \{L, -L, 0\}^T$ in the above equation, we have

$$\vec{H}_{O1} = \begin{Bmatrix} L \\ -L \\ 0 \end{Bmatrix} \times m \begin{Bmatrix} 0 \\ 0 \\ -v_G \end{Bmatrix} = \begin{Bmatrix} mLv_G \\ mLv_G \\ 0 \end{Bmatrix}$$

After Collision:

$$\vec{H}_{O2} = \vec{R}_G \times m(\vec{0} + \vec{\omega}_2 \times \vec{R}_G) + [I_G]\vec{\omega}_2 = [I_O]\vec{\omega}_2$$

in which the moment of inertia matrix about the fixed point O can be determined as

$$[I_O] = \frac{1}{12}m \begin{bmatrix} (2L)^2 & 0 & 0 \\ 0 & (2L)^2 & 0 \\ 0 & 0 & 2(2L)^2 \end{bmatrix} + m \left(2L^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} L^2 & -L^2 & 0 \\ -L^2 & L^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)$$

Now by equating the angular momentum vectors before and after the collision

$$\vec{H}_{O2} = \begin{bmatrix} \frac{4}{3}mL^2 & mL^2 & 0 \\ mL^2 & \frac{4}{3}mL^2 & 0 \\ 0 & 0 & \frac{8}{3}mL^2 \end{bmatrix} \begin{Bmatrix} \omega_{x2} \\ \omega_{y2} \\ \omega_{z2} \end{Bmatrix} = \begin{Bmatrix} \frac{4}{3}mL^2\omega_{x2} + mL^2\omega_{y2} \\ mL^2\omega_{x2} + \frac{4}{3}mL^2\omega_{y2} \\ \frac{8}{3}mL^2\omega_{z2} \end{Bmatrix} = \begin{Bmatrix} mLv_G \\ mLv_G \\ 0 \end{Bmatrix},$$

we have $\omega_{z2} = 0$ and

$$\begin{cases} \frac{4}{3}\omega_{x2} + \omega_{y2} = v_G/L \\ \omega_{x2} + \frac{4}{3}\omega_{y2} = v_G/L \end{cases}.$$

The final result can be written as

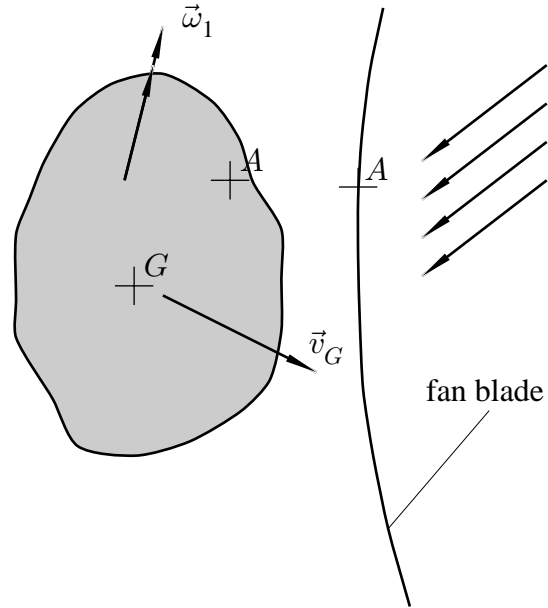
$$\vec{\omega}_2 = \left\{ \frac{3v_G}{7L}, \frac{3v_G}{7L}, 0 \right\}^T$$

Example – When the Shit Hits the Fan

An irregular object with mass 0.4 kg is originally flying with a velocity of $\vec{v}_G = \{4, 3, 5\}^T$ m/sec and with an angular velocity of $\vec{\omega}_1 = \{1, 2, -2\}^T$ rad/sec when it is struck by a fan blade at point A . Point A can be described relative to the center of mass G by position vector $\vec{R}_{G/A} = \{0.2, -0.1, 0.3\}^T$ m. If the velocity of point A of the object is $\{25, 10, 0\}^T$ m/sec at the instant of the impact, find the angular velocity $\vec{\omega}_2$ of the object immediately after the impact.

The moment of inertia matrix $[I_G]$ of the object is given as

$$[I_G] = \begin{bmatrix} 1.3 & -1 & 0.22 \\ -1 & 1.8 & 0.3 \\ 0.22 & 0.3 & 2 \end{bmatrix}$$



SOLUTION:

Since the force of the fan blade is applied at point A , the moment at point A is zero; hence, we can have conservation of angular momentum about point A , i.e.,

$$(\vec{H}_A)_1 = (\vec{H}_A)_2 \quad .$$

Before Impact:

$$\begin{aligned} (\vec{H}_A)_1 &= \vec{R}_{G/A} \times m\vec{v}_G + [I_G]\vec{\omega}_1 \\ &= \begin{Bmatrix} 0.2 \\ -0.1 \\ 0.3 \end{Bmatrix} \times 0.4 \begin{Bmatrix} 4 \\ 3 \\ 5 \end{Bmatrix} + \begin{bmatrix} 1.3 & -1 & 0.22 \\ -1 & 1.8 & 0.3 \\ 0.22 & 0.3 & 2 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \\ -2 \end{Bmatrix} \\ &= \begin{Bmatrix} -0.56 \\ 0.08 \\ 0.40 \end{Bmatrix} + \begin{Bmatrix} -1.14 \\ 2.00 \\ -3.18 \end{Bmatrix} = \begin{Bmatrix} -1.70 \\ 2.08 \\ -2.78 \end{Bmatrix} \end{aligned}$$

After Impact:

Since \vec{v}_A is given, we can write

$$\vec{v}_G = \vec{v}_A + \vec{\omega}_2 \times \vec{R}_{G/A}$$

and the angular momentum about point A after the impact is

$$\begin{aligned}
 (\vec{H}_A)_2 &= \vec{R}_{G/A} \times m(\vec{v}_A + \vec{\omega}_2 \times \vec{R}_{G/A}) + [I_G]\vec{\omega}_2 \\
 &= \vec{R}_{G/A} \times m\vec{v}_A + m(\vec{R}_{G/A} \times \vec{\omega}_2 \times \vec{R}_{G/A}) + [I_G]\vec{\omega}_2 \\
 &= \vec{R}_{G/A} \times m\vec{v}_A + ([I_T] + [I_G])\vec{\omega}_2
 \end{aligned}$$

in which

$$\begin{aligned}
 [I_T] &= 0.4 \left((0.2^2 + (-0.1)^2 + 0.3^2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.04 & -0.02 & 0.06 \\ -0.02 & 0.01 & -0.03 \\ 0.06 & -0.03 & 0.09 \end{bmatrix} \right) \\
 &= 0.4 \begin{bmatrix} 0.10 & 0.02 & -0.06 \\ 0.02 & 0.13 & 0.03 \\ -0.06 & 0.03 & 0.05 \end{bmatrix} = \begin{bmatrix} 0.04 & 0.008 & -0.024 \\ 0.008 & 0.052 & 0.012 \\ -0.024 & 0.012 & 0.02 \end{bmatrix}
 \end{aligned}$$

Therefore, the moment of inertia matrix about point A is calculated as

$$[I_A] = [I_T] + [I_G] = \begin{bmatrix} 1.34 & -0.992 & 0.196 \\ -0.992 & 1.852 & 0.312 \\ 0.196 & 0.312 & 2.02 \end{bmatrix}$$

Equating the two angular momentum terms we have

$$\begin{Bmatrix} -1.70 \\ 2.08 \\ -2.78 \end{Bmatrix} = \begin{Bmatrix} 0.2 \\ -0.1 \\ 0.3 \end{Bmatrix} \times 0.4 \begin{Bmatrix} 25 \\ 10 \\ 0 \end{Bmatrix} + [I_A]\vec{\omega}_2$$

or

$$[I_A]\vec{\omega}_2 = \begin{Bmatrix} -1.70 \\ 2.08 \\ -2.78 \end{Bmatrix} - \begin{Bmatrix} -1.2 \\ 3.0 \\ 1.8 \end{Bmatrix}$$

and the 3×3 matrix equation

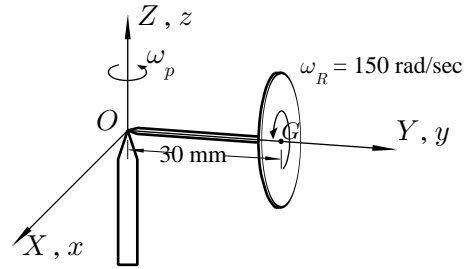
$$\begin{bmatrix} 1.34 & -0.992 & 0.196 \\ -0.992 & 1.852 & 0.312 \\ 0.196 & 0.312 & 2.02 \end{bmatrix} \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix}_2 = \begin{Bmatrix} -0.5 \\ -0.92 \\ -4.58 \end{Bmatrix}$$

yields

$$\begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix}_2 = \begin{Bmatrix} -0.2374 \\ -0.2524 \\ -2.2053 \end{Bmatrix}$$

Example – Simple Gyroscopic Motion

The toy gyroscope consists of a rotor R which is attached to the frame of negligible mass. If the rotor is spinning about its axle at an angular speed $\omega_R = 150$ rad/sec, determine the constant angular velocity ω_p at which the frame is precessing about the pivot point at O . OG moves in the horizontal plane. The rotor has a mass of 200 g and a radius of gyration $k_{OG} = 20$ mm about OG .



Solution:

Given: $I_{yy} = (200)(20)^2 = 8 \times 10^4$ g-mm². I_{xx} and I_{zz} were not given for this particular problem, so with $\vec{R}_G = \{0, 30, 0\}^T$

$$[I_O] = [I_G] + [I_T] = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & 8 \times 10^4 & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} + 200 \left(900 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 900 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)$$

$$= \begin{bmatrix} I_{xx} + 180000 & 0 & 0 \\ 0 & 80000 & 0 \\ 0 & 0 & I_{zz} + 180000 \end{bmatrix}$$

Now we can calculate the angular momentum about point O as

$$\{H_O\}_{xyz} = [I_O] \begin{Bmatrix} 0 \\ 150 \\ \omega_p \end{Bmatrix}_{xyz} = \begin{Bmatrix} 0 \\ 1.2 \times 10^7 \\ (I_{zz} + 180000)\omega_p \end{Bmatrix}_{xyz} .$$

The angular velocity of the xyz coordinate system is $\vec{\Omega} = \{0, 0, \omega_p\}^T$ even though the angular velocity of the rotor is $\vec{\omega}_{xyz} = \{0, 150, \omega_p\}^T$. With $\vec{\Omega}$, we can calculate the rate of change of angular momentum as

$$\left\{ \sum M_O \right\}_{xyz} = \frac{d_c}{dt} \{H_O\}_{xyz} + \vec{\Omega} \times \{H_O\}_{xyz} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ \omega_p \end{Bmatrix} \times \begin{Bmatrix} 0 \\ 1.2 \times 10^7 \\ (I_{zz} + 180000)\omega_p \end{Bmatrix} .$$

Now by calculating the moment of the rotor's weight about point O , we have

$$\left\{ \sum M_O \right\}_{xyz} = \vec{r} \times \vec{F} = \begin{Bmatrix} 0 \\ 30 \\ 0 \end{Bmatrix} \times \begin{Bmatrix} 0 \\ 0 \\ (-200)(9810) \end{Bmatrix} = \begin{Bmatrix} -5.89 \times 10^7 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -1.2 \times 10^7 \omega_p \\ 0 \\ 0 \end{Bmatrix}$$

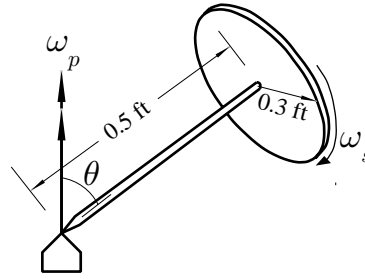
or

$$\omega_p = 4.91 \text{ rad/sec} .$$

As expected, I_{xx} and I_{zz} did not enter into the results of this particular problem. But for more complicated gyroscopic motion problems, these other moment of inertia terms will also be needed.

Example – More Complicated Gyroscopic Motion

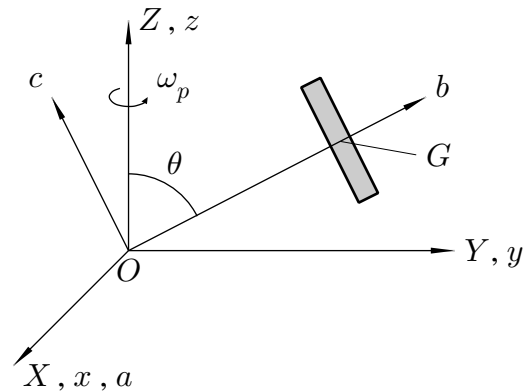
The top consists of a thin disk that has a weight of 8 lb and a radius of 0.3 ft. The rod has a negligible mass and a length of 0.5 ft. If the top is spinning with an angular velocity $\omega_s = 300$ rad/sec, determine the steady-state precessional angular velocity ω_p of the rod as a function of θ . Do specific examples for $\theta = 40^\circ$ and 90° .



Solution:

It is convenient to solve this problem with three different coordinate systems: XYZ as the fixed system, xyz as the rotating system which follows the top and abc as the system for convenience in determining the moment of inertia matrix of the disc. Since this is a steady-state precession problem, it is easiest to solve the problem at the instant when the xyz and XYZ systems are aligned, i.e., when the transformation between the two systems is defined as

$$\begin{Bmatrix} q_X \\ q_Y \\ q_Z \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} q_x \\ q_y \\ q_z \end{Bmatrix} .$$



The transformation between the xyz and the abc system can also be established as

$$\begin{Bmatrix} q_x \\ q_y \\ q_z \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\frac{\pi}{2} - \theta) & \cos(\pi + \theta) \\ 0 & \cos \theta & \cos(\frac{\pi}{2} - \theta) \end{bmatrix} \begin{Bmatrix} q_a \\ q_b \\ q_c \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin \theta & -\cos \theta \\ 0 & \cos \theta & \sin \theta \end{bmatrix} \begin{Bmatrix} q_a \\ q_b \\ q_c \end{Bmatrix} .$$

With the circular disk being a simple geometry in the abc coordinate system, it is easy to determine that

$$[I_G]_{abc} = \left(\frac{8}{32.2} \right) (0.3)^2 \begin{bmatrix} 0.25 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.25 \end{bmatrix} = \begin{bmatrix} 0.00559 & 0 & 0 \\ 0 & 0.01118 & 0 \\ 0 & 0 & 0.00559 \end{bmatrix}_{abc} ,$$

now the parallel-axis theorem can be applied by defining $\vec{R}_G = \{0, 0.5, 0\}^T$ and

$$[I_T]_{abc} = \left(\frac{8}{32.2} \right) \left(0.25 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0.06211 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.06211 \end{bmatrix}_{abc} .$$

The moment of inertia matrix of the disk about point O can now be calculated as

$$[I_O]_{abc} = [I_G]_{abc} + [I_T]_{abc} = \begin{bmatrix} 0.0677 & 0 & 0 \\ 0 & 0.01118 & 0 \\ 0 & 0 & 0.0677 \end{bmatrix}_{abc} .$$

To determine the angular momentum of the disk, use the abc coordinate system. The angular velocity of the disk in the abc system can be determined via one transformation as

$$\{\omega\}_{abc} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin \theta & \cos \theta \\ 0 & -\cos \theta & \sin \theta \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \omega_p \end{Bmatrix}_{xyz} + \begin{Bmatrix} 0 \\ 300 \\ 0 \end{Bmatrix}_{abc} = \begin{Bmatrix} 0 \\ \omega_p \cos \theta + 300 \\ \omega_p \sin \theta \end{Bmatrix}_{abc},$$

and it follows that the angular momentum in the abc system is simply

$$\{H_O\}_{abc} = [I_O]_{abc} \{\omega\}_{abc} = \begin{Bmatrix} 0 \\ 0.01118(\omega_p \cos \theta + 300) \\ 0.0677\omega_p \sin \theta \end{Bmatrix}_{abc}.$$

To calculate the rate of change of angular momentum in a rotating coordinate system, rewrite the angular velocity vector $\vec{\Omega} = \{0, 0, \omega_p\}_{xyz}^T$ in the abc system via a transformation as $\vec{\Omega} = \{0, \omega_p \cos \theta, \omega_p \sin \theta\}_{abc}^T$ and then

$$\begin{aligned} \frac{d}{dt}\{H_O\} &= \frac{d_c}{dt}\{H_O\} + \vec{\Omega} \times \{H_O\} = \vec{0} + \begin{Bmatrix} 0 \\ \omega_p \cos \theta \\ \omega_p \sin \theta \end{Bmatrix}_{abc} \times \begin{Bmatrix} 0 \\ 0.01118(\omega_p \cos \theta + 300) \\ 0.0677\omega_p \sin \theta \end{Bmatrix}_{abc} \\ &= \begin{Bmatrix} 0.0677\omega_p^2 \sin \theta \cos \theta - 0.01118\omega_p^2 \sin \theta \cos \theta - 3.354\omega_p \sin \theta \\ 0 \\ 0 \end{Bmatrix}_{abc}. \end{aligned}$$

Now after the transformation to the xyz system

$$\left\{ \sum M_O \right\}_{xyz} = \frac{d}{dt}\{H_O\}_{xyz} = \begin{Bmatrix} 0.05652\omega_p^2 \sin \theta \cos \theta - 3.354\omega_p \sin \theta \\ 0 \\ 0 \end{Bmatrix}_{xyz}.$$

To determine the sum of moments about point O due to the weight of the top, use the cross product

$$\left\{ \sum M_O \right\}_{xyz} = \vec{r} \times \vec{F} = \begin{Bmatrix} 0 \\ 0.5 \sin \theta \\ 0.5 \cos \theta \end{Bmatrix}_{xyz} \times \begin{Bmatrix} 0 \\ 0 \\ -8 \end{Bmatrix}_{xyz} = \begin{Bmatrix} -4 \sin \theta \\ 0 \\ 0 \end{Bmatrix}_{xyz}$$

and by equating the above two equations, we have a quadratic equation for ω_p :

$$0.05652\omega_p^2 \cos \theta - 3.354\omega_p + 4 = 0$$

Case One: At $\theta = 40^\circ$, the quadratic equation $0.0432968\omega_p^2 - 3.354\omega_p + 4 = 0$ yields two roots: $\omega_p = 1.21$ rad/sec as the *low precession* angular velocity and $\omega_p = 76.25$ rad/sec as the *high precession* angular velocity.

Case Two: At $\theta = 90^\circ$, the quadratic equation reduces to a linear equation $-3.354\omega_p + 4 = 0$ which yields $\omega_p = 1.193$ rad/sec as the only precession angular velocity for this special case.