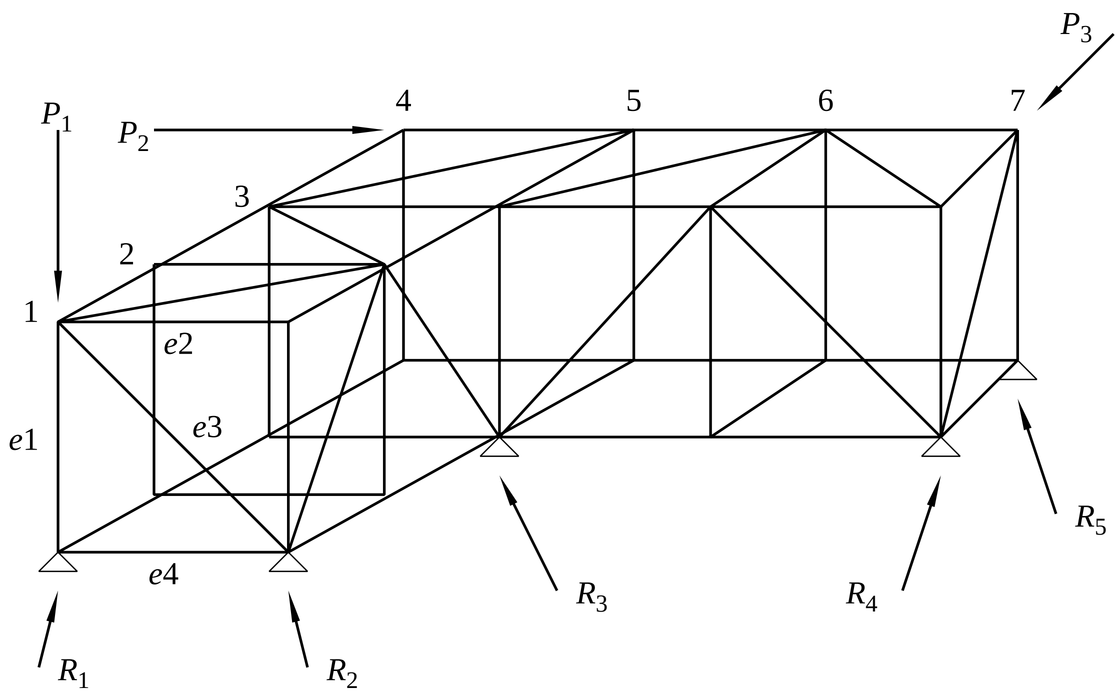
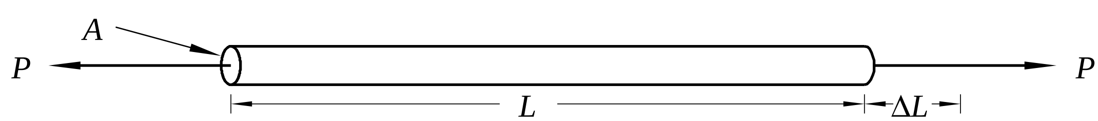
Matrix Analysis of Trusses

This formulation is applicable for both statically determinant and statically indeterminant problems. A statically determinant problem is one which can be solved using the equilibrium equations alone while a statically indeterminant problem needs the constitutive equation for the material and displacement compatibility in addition to the equilibrium equations.



The Element Stiffness Matrix

The basic element of space trusses is a “pinned rod element” as shown below:



Consider a one-dimensional rod element with area A, length L and Young’s modulus E. Hooke’s Law implies

$$\sigma = E\epsilon \qquad , \tag{1}$$

where $\sigma = P/A$ is the stress and $\epsilon = \Delta L/L$ is the strain. Hence

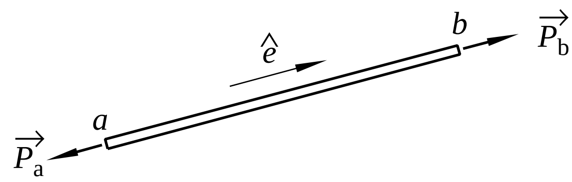
$$\frac{P}{A} = E\frac{\Delta L}{L} \qquad , \tag{2}$$

or the relationship between P and the relative displacement ΔL can be written as

$$\boxed{P = \left(\frac{EA}{L}\right)\Delta L \quad .} \tag{3}$$

The above relationship is one-dimensional, so in order to utilize it in a three-dimensional space truss, some coordinate transformations must be applied.

Consider a rod element as shown to the right. If a tensile force P is applied to its two nodes, then the force vectors at nodes a and b are $\vec{P}_a = -P\hat{e}$ and $\vec{P}_b = P\hat{e}$, respectively. The unit vector $\hat{e}$ is defined from node a toward node b.



All the nodal forces of the element can be grouped into a 6×1 vector $\vec{f}$ defined as

$$\vec{f} = \left\{ \begin{array}{c} \vec{P}_a \\ \vec{P}_b \end{array} \right\} = P \left\{ \begin{array}{c} -\hat{e} \\ \hat{e} \end{array} \right\} \quad . \tag{4}$$

Define also a 6×1 element displacement vector $\vec{\delta}$ as

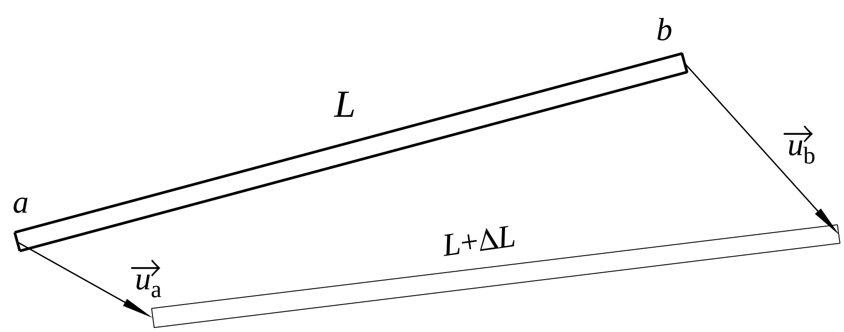
$$\vec{\delta} = \left\{ \begin{array}{c} \vec{u}_a \\ \vec{u}_b \end{array} \right\} \quad , \tag{5}$$

where $\vec{u}_a$ and $\vec{u}_b$ are, respectively, the 3×1 displacement vectors at nodes a and b measured from the undeformed state.

We can now introduce an element stiffness matrix, $[k_e]$, which correlates the nodal forces and the nodal displacements for the rod element as

$$\{\vec{f}\} = [k_e]\{\vec{\delta}\} \qquad \text{or} \qquad \left\{ \begin{array}{c} \vec{P}_a \\ \vec{P}_b \end{array} \right\} = [k_e] \left\{ \begin{array}{c} \vec{u}_a \\ \vec{u}_b \end{array} \right\} \quad . \tag{6}$$

Consider the rod element at its “before stretch” and “after stretch” positions as shown:



Lets for derivation purposes define a local coordinate system with the origin located at point a of the element. Then the position vectors of the end points a and b “before” and “after” the deformation can be tabulated as shown below:

	Before Stretch	After Stretch
Node a	$\vec{0}$	$\vec{u}_a$
Node b	$\vec{L} = L\hat{e}$	$\vec{L} + \vec{u}_b$
Length of rod	$\lvert\vec{L}\rvert = L$	$\lvert(\vec{L} + \vec{u}_b) - \vec{u}_a\rvert = L + \Delta L$

To simplify the expression for $L + \Delta L$, lets rearrange the terms as follows:

$$\begin{aligned} L + \Delta L &= |(\vec{L} + \vec{u}_b) - \vec{u}_a| = |(\vec{u}_b - \vec{u}_a) + \vec{L}| \\ &= \sqrt{[(\vec{u}_b - \vec{u}_a) + \vec{L}] \cdot [(\vec{u}_b - \vec{u}_a) + \vec{L}]} \\ &= \sqrt{(\vec{u}_b - \vec{u}_a) \cdot (\vec{u}_b - \vec{u}_a) + 2\vec{L} \cdot (\vec{u}_b - \vec{u}_a) + \vec{L} \cdot \vec{L}} \\ &= L\sqrt{\left(\frac{|\vec{u}_b - \vec{u}_a|}{L}\right)^2 + 1 + \frac{2\vec{L}}{L} \cdot \left(\frac{\vec{u}_b - \vec{u}_a}{L}\right)} \quad . \end{aligned} \tag{7}$$

Since for small strain

$$\left(\frac{|\vec{u}_b - \vec{u}_a|}{L}\right)^2 << \left(\frac{|\vec{u}_b - \vec{u}_a|}{L}\right) \quad , \tag{8}$$

it is clear that

$$\begin{aligned} L + \Delta L &\approx L\sqrt{1 + 2\hat{e} \cdot \left(\frac{\vec{u}_b - \vec{u}_a}{L}\right)} \\ &\approx L\left(1 + \hat{e} \cdot (\vec{u}_b - \vec{u}_a)/L + \ldots\right) \\ &= L + \hat{e} \cdot (\vec{u}_b - \vec{u}_a) \end{aligned} \tag{9}$$

or

$$\Delta L = \hat{e} \cdot (\vec{u}_b - \vec{u}_a) = \{ -\hat{e}^T \quad \hat{e}^T \} \left\{ \begin{array}{c} \vec{u}_a \\ \vec{u}_b \end{array} \right\} \quad . \tag{10}$$

We are now in position to use the one-dimensional result in equation (3). Recall from equation (4)

$$\{\vec{f}\} = \left\{ \begin{array}{c} \vec{P}_a \\ \vec{P}_b \end{array} \right\} = P \left\{ \begin{array}{c} -\hat{e} \\ \hat{e} \end{array} \right\} \quad , \tag{11}$$

we can substitute $EA\Delta L/L$ for P and obtain

$$\{\vec{f}\} = \frac{EA}{L} \left\{ \begin{array}{c} -\hat{e} \\ \hat{e} \end{array} \right\} \Delta L = \frac{EA}{L} \left\{ \begin{array}{c} -\hat{e} \\ \hat{e} \end{array} \right\} \{ -\hat{e}^T \quad \hat{e}^T \} \left\{ \begin{array}{c} \vec{u}_a \\ \vec{u}_b \end{array} \right\} \tag{12}$$

Using equation (10) and after performing the matrix product, the force-displacement relationship becomes

$$\{\vec{f}\} = \frac{EA}{L} \begin{bmatrix} \hat{e}\hat{e}^T & -\hat{e}\hat{e}^T \\ -\hat{e}\hat{e}^T & \hat{e}\hat{e}^T \end{bmatrix} \{\vec{\delta}\} \quad . \tag{13}$$

By comparing equations (13) and (6), the element stiffness matrix $[k_e]$ can be written by inspection as

$$\boxed{[k_e] = \frac{EA}{L} \begin{bmatrix} \hat{e}\hat{e}^T & -\hat{e}\hat{e}^T \\ -\hat{e}\hat{e}^T & \hat{e}\hat{e}^T \end{bmatrix} \quad .} \tag{14}$$

$[k_e]$ has a dimension of 6×6. Note: $\hat{e}\hat{e}^T$, which is 3×3, is not the same as $\hat{e}^T\hat{e}$, the scalar product. For example, if

$$\hat{e} = \left\{ \begin{array}{c} 1 \\ 2 \\ -1 \end{array} \right\}, \qquad \text{then} \qquad \hat{e}^T\hat{e} = \{ 1 \quad 2 \quad -1 \} \left\{ \begin{array}{c} 1 \\ 2 \\ -1 \end{array} \right\} = (1)^2 + (2)^2 + (-1)^2 = 6,$$

is a scalar, but

$$\hat{e}\hat{e}^T = \left\{ \begin{array}{c} 1 \\ 2 \\ -1 \end{array} \right\} \{ 1 \quad 2 \quad -1 \} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{bmatrix}$$

is a 3×3 square and symmetric matrix.

Assembling Elements to Form a Structure

For a typical space truss as shown in an earlier figure, the joints will be numbered from 1 to N, these numbers will be referred to as the "Global Node Numbers." The elements, which are pinned-rod elements, are to be numbered as $e1$, $e2$, ..., and eM, where M is the total number of rod elements used in the structure. The assembly process requires the following information:

(1) The xyz coordinates of the Global nodes.

(2) The properties: Young's modulus E_i and cross-sectional area A_i of all the elements.

(3) The way the elements are to be connected to the global nodes, e.g., nodes a and b of element i are connected to global nodes m and n of the structure, respectively. Once the connections are defined, the length L_i can be calculated readily from the coordinates of the global nodes.

(4) The conditions of *where* (which global nodes) and *how* (e.g., completely restricted or restricted only in one direction) the structure is supported. These are called the "boundary conditions."

(5) The loading conditions of the structure. Concentrated loads may be applied at the joints of the space truss.

Displacement Compatibility

The nodal displacement $\vec{u}_a$ and $\vec{u}_b$ of element i, once it is connected to global nodes m and n, must be equal to the displacements of all other elements which have one of their two nodes attached there. This constitutes the compatibility of displacements. At the supports, the displacements are restricted to zero; these are called boundary conditions.

Mathematically, it is convenient to define a mapping matrix $[\beta_i]$ for each element so that local nodal displacements, $\vec{\delta}_i$ for element i, can be linked to global displacements, $\vec{U}$, as

$$\vec{\delta}_i = [\beta]_i \vec{U} \qquad , \tag{15}$$

or in matrix form

$$\left\{ \begin{matrix} \vec{u}_a \\ \vec{u}_b \end{matrix} \right\} = \begin{bmatrix} [0] & \cdots & \cdots & [I] & \cdots & [0] & \cdots & \cdots & [0] \\ [0] & \cdots & \cdots & [0] & \cdots & [I] & \cdots & \cdots & [0] \end{bmatrix} \left\{ \begin{matrix} \vec{U}_1 \\ \vdots \\ \vdots \\ \vec{U}_m \\ \vdots \\ \vec{U}_n \\ \vdots \\ \vdots \\ \vec{U}_N \end{matrix} \right\} \qquad . \tag{16}$$

The $6 \times 3N$ matrix $[\beta]_i$ contains mostly zeroes as indicated by the great number of 3×3 zero matrices $[0]$. The two 3×3 identity matrices $[I]$, are located on columns m and n in rows 1 and 2, respectively. In the computer, the matrix multiplcation in equation (16) is never actually performed because of the fruitless task of multiplying zeroes. Instead, pointers are used to locate the nonzero entries.

Equilibrium Conditions

For a static analysis, the equilibrium conditions can be satisfied by enforcing Newton's First Law. It requires that the sum of the force vectors from (i) the elements, (ii) the applied forces and (iii) the reaction forces, must be equal to zero at every global nodes. In this section, the contribution from each element will be determined.

To map the contribution of the *element forces,* $\vec{P}_a$ and $\vec{P}_b$, of element i to the total global *nodal forces* $\vec{F}_m$ and $\vec{F}_n$ at the nodes m and n, we can again use the matrix $[\beta_i]$ as

$$\left\{ \begin{array}{c} \vec{F}_1 \\ \vdots \\ \vdots \\ \vec{F}_m \\ \vdots \\ \vec{F}_n \\ \vdots \\ \vdots \\ \vec{F}_N \end{array} \right\}_i = - \begin{bmatrix} [0] & [0] \\ \vdots & \vdots \\ \vdots & \vdots \\ [I] & [0] \\ \vdots & \vdots \\ [0] & [I] \\ \vdots & \vdots \\ \vdots & \vdots \\ [0] & [0] \end{bmatrix} \left\{ \begin{array}{c} \vec{P}_a \\ \vec{P}_b \end{array} \right\}_i \tag{17}$$

or simply

$$\vec{F}_i = -[\beta]_i^T \, \vec{f}_i \qquad , \tag{18}$$

where $\vec{F}_i$ represents the contribution from element i to the total global force vector $\vec{F}$.
Since for each element, the force-displacement relationship can be written as

$$\vec{f}_i = [k_e]_i \, \vec{\delta}_i \qquad , \tag{19}$$

its substitution into equation (18) with the aide of equation (15) yields

$$\vec{F}_i = -[\beta]_i^T \, [k_e]_i \, \vec{\delta}_i = -[\beta]_i^T \, [k_e]_i \, [\beta]_i \, \vec{U} \qquad , \tag{20}$$

or

$$\vec{F}_i = -[K_e]_i \, \vec{U} \qquad , \tag{21}$$

where

$$[K_e]_i = [\beta]_i^T \, [k_e]_i \, [\beta]_i \tag{22}$$

is the expanded stiffness matrix of element i using a global numbering system. As expected, $[K_e]_i$ contains mostly zeroes because it affects only 2 of the N nodes.

The Global Stiffness Matrix

To complete the picture for the entire truss, Newton's First Law can be imposed for the N global nodes ($3N$ degrees-of-freedom) as

$$\sum_{i=1}^{M} \vec{F}_i \; + \; \vec{F}' \; = \; \vec{0} \qquad , \tag{23}$$

where $\vec{F}'$ is the external force vector which contains the applied and reaction forces. Substituting equation (21) into (23), Newton's First Law can be written in term of the global displacement vector $\vec{U}$ as

$$\vec{F}' = -\sum_{i=1}^{M} \vec{F}_i = \left(\sum_{i=1}^{M} [K_e]_i \right) \vec{U} \qquad . \tag{24}$$

Define now the global stiffness matrix $[K]$ as

$$[K] = \sum_{i=1}^{M} [K_e]_i \qquad , \tag{25}$$

then equation (24) can be written as a matrix equation in the form

$$[K]\,\vec{U} = \vec{F}' \qquad , \tag{26}$$

where the displacement vector $\vec{U}$ is usually the unknown quantity.

Boundary Conditions

A fraction of the N global nodes must be fixed to ensure stability of the truss. These we shall call boundary nodes. It is possible to order the global nodes in a manner such that the global displacement vector can be partitioned as

$$\vec{U} = \left\{ \begin{array}{c} \vec{U}_f \\ \vec{U}_r \end{array} \right\} = \left\{ \begin{array}{c} \vec{U}_f \\ \vec{0} \end{array} \right\} \qquad , \tag{27}$$

where $\vec{U}_f$ contains the unknown displacements at the "free nodes" and $\vec{U}_r = \vec{0}$ contains the displacements at the "restricted nodes."

Similarly, the global force vector $\vec{F}$ can be partitioned as

$$\vec{F} = \left\{ \begin{array}{c} \vec{F}_f \\ \vec{F}_r \end{array} \right\} = \left\{ \begin{array}{c} \vec{F}_f \\ \vec{R} \end{array} \right\} \qquad , \tag{28}$$

where $\vec{F}_f$ contains the total forces at the free nodes and $\vec{R}$ contains the reaction forces at the restricted nodes. The renumbering of nodes also causes the global stiffness matrix $[K]$ to be partitioned as

$$\left\{ \begin{array}{c} \vec{F}_f \\ \vec{R} \end{array} \right\} = \begin{bmatrix} K_{ff} & K_{fr} \\ K_{rf} & K_{rr} \end{bmatrix} \left\{ \begin{array}{c} \vec{U}_f \\ \vec{0} \end{array} \right\} \qquad . \tag{29}$$

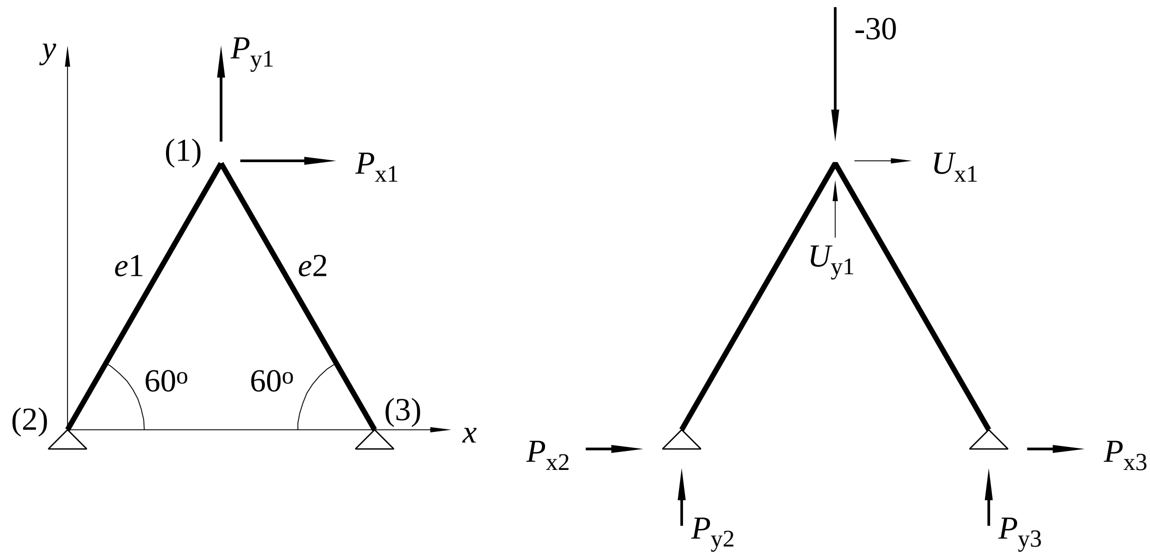
In the above equation, the displacement vector $\vec{U}_f$ is unknown and the reaction force vector $\vec{R}$ is directly related to $\vec{U}$. The solution can be obtained first solving the top partitioned matrix equation, i.e.,

$$\left\{\vec{F}_f\right\} = [K_{ff}]\left\{\vec{U}_f\right\} \qquad \text{or} \qquad \left\{\vec{U}_f\right\} = [K_{ff}]^{-1}\left\{\vec{F}_f\right\} \qquad , \tag{30}$$

and then calculate the reaction forces with the bottom partitioned matrix equation as

$$\left\{\vec{R}\right\} = [K_{rf}]\left\{\vec{U}_f\right\} \qquad . \tag{31}$$

Example 1 – A Two-Element Truss



Parameters: Number of Nodes $N = 3$, Number of elements $M = 2$, element properties: $EA/L = 1$ for both elements

<u>Element Number $e1$:</u> $\hat{e}^T = \{\cos 60^\circ\ ,\ \sin 60^\circ\}$

$$[k_e]_1 = \frac{EA}{L}\begin{Bmatrix} -0.500 \\ -0.866 \\ 0.500 \\ 0.866 \end{Bmatrix}\{\, -0.500 \quad -0.866 \quad 0.500 \quad 0.866 \,\}$$

$$= \begin{bmatrix} 0.250 & 0.433 & -0.250 & -0.433 \\ 0.433 & 0.750 & -0.433 & -0.750 \\ -0.250 & -0.433 & 0.250 & 0.433 \\ -0.433 & -0.750 & 0.433 & 0.750 \end{bmatrix}$$

Mapping Matrix: local node $a =$ global node (2), local node $b =$ global node (1). Hence,

$$[\beta]_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Expanded element stiffness matrix:

$$[K_e]_1 = [\beta]_1^T [k_e]_1 [\beta]_1 = \begin{bmatrix} 0.250 & 0.433 & -0.250 & -0.433 & 0 & 0 \\ 0.433 & 0.750 & -0.433 & -0.750 & 0 & 0 \\ -0.250 & -0.433 & 0.250 & 0.433 & 0 & 0 \\ -0.433 & -0.750 & 0.433 & 0.750 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Element Number $e2$: $\hat{e}^T = \{\cos 120^\circ\ ,\ \sin 120^\circ\}$

$$[k_e]_2 = \frac{EA}{L}\begin{Bmatrix} 0.500 \\ -0.866 \\ -0.500 \\ 0.866 \end{Bmatrix}\{\,0.500 \quad -0.866 \quad -0.500 \quad 0.866\,\}$$

$$= \begin{bmatrix} 0.250 & -0.433 & -0.250 & 0.433 \\ -0.433 & 0.750 & 0.433 & -0.750 \\ -0.250 & 0.433 & 0.250 & -0.433 \\ 0.433 & -0.750 & -0.433 & 0.750 \end{bmatrix}$$

Mapping Matrix: local node a = global node (3), local node b = global node (1). Hence,

$$[\beta]_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Expanded element stiffness matrix:

$$[K_e]_2 = [\beta]_2^T[k_e]_2[\beta]_2 = \begin{bmatrix} 0.250 & -0.433 & 0 & 0 & -0.250 & 0.433 \\ -0.433 & 0.750 & 0 & 0 & 0.433 & -0.750 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.250 & 0.433 & 0 & 0 & 0.250 & -0.433 \\ 0.433 & -0.750 & 0 & 0 & -0.433 & 0.750 \end{bmatrix}$$

TOTAL STIFFNESS MATRIX:

$$[K] = [K_e]_1 + [K_e]_2 = \begin{bmatrix} 0.500 & 0 & -0.250 & -0.433 & -0.250 & 0.433 \\ 0 & 1.500 & -0.433 & -0.750 & 0.433 & -0.750 \\ -0.250 & -0.433 & 0.250 & 0.433 & 0 & 0 \\ -0.433 & -0.750 & 0.433 & 0.750 & 0 & 0 \\ -0.250 & 0.433 & 0 & 0 & 0.250 & -0.433 \\ 0.433 & -0.750 & 0 & 0 & -0.433 & 0.750 \end{bmatrix}$$

EQUILIBRIUM EQUATION:

Consider the loading conditions: $P_{x1} = 0$ and $P_{y1} = -30$. Global Nodes (2) and (3) are fixed. The equilibrium equation is therefore

$$\begin{Bmatrix} 0 \\ -30 \\ P_{x2} \\ P_{y2} \\ P_{x3} \\ P_{y3} \end{Bmatrix} = \begin{bmatrix} 0.500 & 0 & -0.250 & -0.433 & -0.250 & 0.433 \\ 0 & 1.500 & -0.433 & -0.750 & 0.433 & -0.750 \\ -0.250 & -0.433 & 0.250 & 0.433 & 0 & 0 \\ -0.433 & -0.750 & 0.433 & 0.750 & 0 & 0 \\ -0.250 & 0.433 & 0 & 0 & 0.250 & -0.433 \\ 0.433 & -0.750 & 0 & 0 & -0.433 & 0.750 \end{bmatrix}\begin{Bmatrix} U_{x1} \\ U_{y1} \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Since U_{x1} and U_{y1} are the only unknown displacements, the above matrix equation can be simplified by discarding the matrix elements which are multiplied by the zero displacement at the fixed nodes, i.e.,

$$\left\{\begin{array}{c} 0 \\ -30 \\ \hline P_{x2} \\ P_{y2} \\ \\ P_{x3} \\ P_{y3} \end{array}\right\} = \left[\begin{array}{cc} 0.500 & 0 \\ 0 & 1.500 \\ \hline -0.250 & -0.433 \\ -0.433 & -0.750 \\ \\ -0.250 & 0.433 \\ 0.433 & -0.750 \end{array}\right] \left\{\begin{array}{c} U_{x1} \\ U_{y1} \end{array}\right\}$$

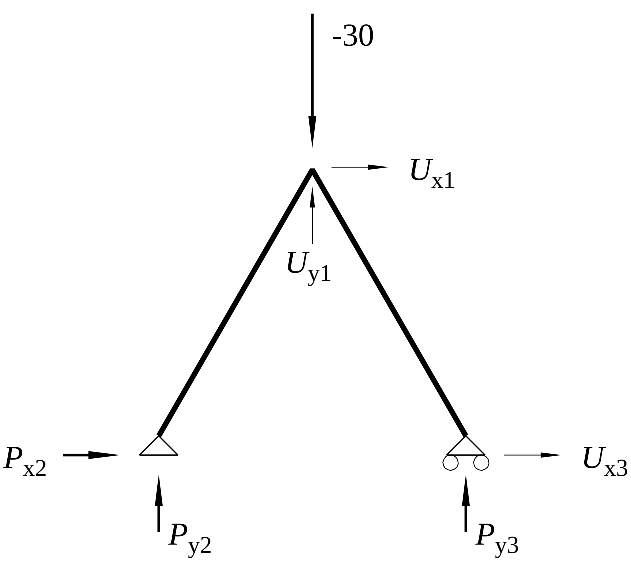
Solve now the top partitioned matrix equation as follows:

$$\left\{\begin{array}{c} 0 \\ -30 \end{array}\right\} = \left[\begin{array}{cc} 0.50 & 0 \\ 0 & 1.50 \end{array}\right] \left\{\begin{array}{c} U_{x1} \\ U_{y1} \end{array}\right\} \Longrightarrow \left\{\begin{array}{c} U_{x1} \\ U_{y1} \end{array}\right\} = \left[\begin{array}{cc} 0.50 & 0 \\ 0 & 1.50 \end{array}\right]^{-1} \left\{\begin{array}{c} 0 \\ -30 \end{array}\right\} = \left\{\begin{array}{c} 0 \\ -20 \end{array}\right\} ,$$

then calculate the reaction forces using the bottom partition as

$$\left\{\begin{array}{c} P_{x2} \\ P_{y2} \\ \\ P_{x3} \\ P_{y3} \end{array}\right\} = \left[\begin{array}{cc} -0.250 & -0.433 \\ -0.433 & -0.750 \\ \\ -0.250 & 0.433 \\ 0.433 & -0.750 \end{array}\right] \left\{\begin{array}{c} 0 \\ -20 \end{array}\right\} = \left\{\begin{array}{c} 8.66 \\ 15 \\ \\ -8.66 \\ 15 \end{array}\right\}$$

One interesting note resulting from this simple numerical example is that the matrix to be used for determining the unknown displacements can be obtained by eliminating all the columns and rows assigned to the fixed nodal degrees-of-freedom from global matrix.



Unstable Boundary Conditions

Consider a similar case now with the identical loading condition but global node (3) is on a roller in the x-direction, hence, $U_{x2} = U_{y2} = U_{y3} = 0$, but U_{x3} is left as an unknown. We know physically that this is an unstable system, but lets test it numerically nonethesless. By eliminating the rows and columns associated with degrees-of-freedom U_{x2}, U_{y2} and U_{y3}, the equilibrium equation yields

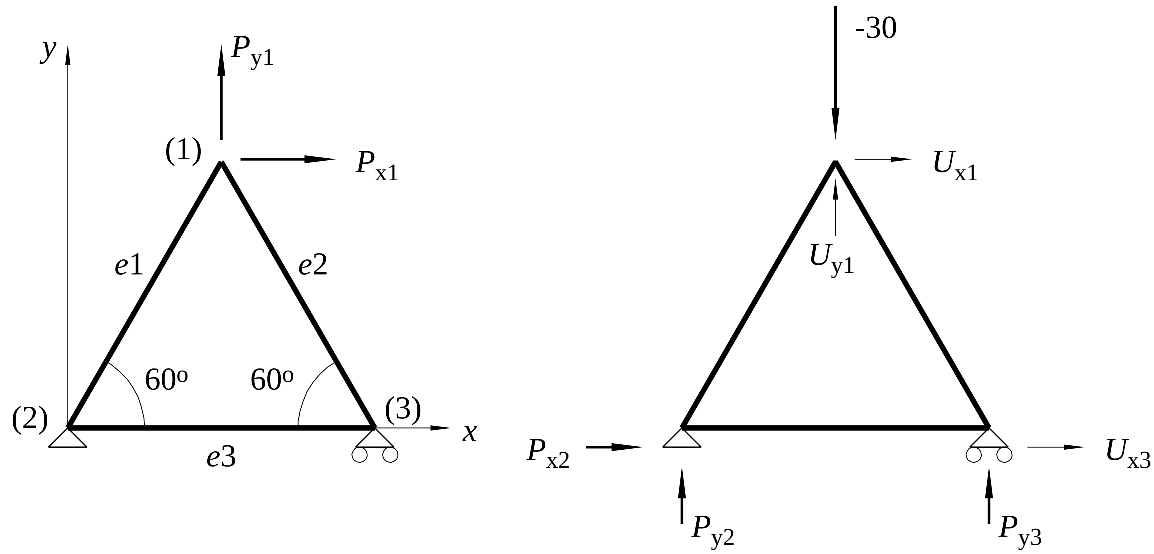
$$\left\{\begin{array}{c} 0 \\ -30 \\ 0 \end{array}\right\} = \left[\begin{array}{ccc} 0.500 & 0 & -0.250 \\ 0 & 1.500 & 0.433 \\ -0.250 & 0.433 & 0.250 \end{array}\right] \left\{\begin{array}{c} U_{x1} \\ U_{y1} \\ U_{x3} \end{array}\right\}$$

It can be shown easily that the matrix equation above is singular by calculating the value of the determinant Δ as

$$\Delta = (0.5)(1.5)(0.25) + 0 + 0 - (-0.25)(1.5)(-0.25) - (0.5)(0.433)(0.433) - 0 = 0 \quad .$$

Example 2 – A Three-Element Truss

To stablize the second structure in the previous example, add another element $e3$ between global nodes (2) and (3).



Parameters: Number of Nodes $N = 3$, Number of elements $M = 3$, element properties: $EA/L = 1$ for all three elements

<u>Element Number $e3$:</u> $\hat{e}^T = \{\cos 0^\circ\ ,\ \sin 0^\circ\}$

$$[k_e]_3 = \frac{EA}{L} \begin{Bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{Bmatrix} \{-1 \quad 0 \quad 1 \quad 0\}$$

$$= \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Mapping Matrix: local node $a =$ global node (2), local node $b =$ global node (3). Hence,

$$[\beta]_3 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Expanded element stiffness matrix:

$$[K_e]_3 = [\beta]_3^T [k_e]_3 [\beta]_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

With the element $e3$ added, the global stiffness matrix becomes

$$[K] = \sum_{m=1}^{3} [K_e]_m = \begin{bmatrix} 0.500 & 0 & -0.250 & -0.433 & -0.250 & 0.433 \\ 0 & 1.500 & -0.433 & -0.750 & 0.433 & -0.750 \\ -0.250 & -0.433 & 1.250 & 0.433 & -1 & 0 \\ -0.433 & -0.750 & 0.433 & 0.750 & 0 & 0 \\ -0.250 & 0.433 & -1 & 0 & 1.250 & -0.433 \\ 0.433 & -0.750 & 0 & 0 & -0.433 & 0.750 \end{bmatrix}$$

It clear that nothing will change if global nodes (2) and (3) are both fixed completely. In this case, only degrees-of-freedom 1 and 2 are free and the matrix $[K_{ff}]$ for the determination of the unknown displacement is again simply

$$[K_{ff}] = \begin{bmatrix} 0.5 & 0 \\ 0 & 1.5 \end{bmatrix}$$

But if global node (3) is on rollers, then degrees-of-freedom 1, 2 and 5 are all free and the equilibrium equation would be

$$\begin{Bmatrix} 0 \\ -30 \\ 0 \end{Bmatrix} = \begin{bmatrix} 0.5 & 0 & -0.25 \\ 0 & 1.5 & 0.433 \\ -0.25 & 0.433 & 1.25 \end{bmatrix} \begin{Bmatrix} U_{x1} \\ U_{y1} \\ U_{x3} \end{Bmatrix}$$

and the solution is

$$\begin{Bmatrix} U_{x1} \\ U_{y1} \\ U_{x3} \end{Bmatrix} = \begin{Bmatrix} 4.33 \\ -22.5 \\ 8.66 \end{Bmatrix} \quad .$$

The reaction forces can be calculated from the new solution as

$$\begin{Bmatrix} P_{x2} \\ P_{y2} \\ P_{y3} \end{Bmatrix} = \begin{bmatrix} -0.250 & -0.433 & -1.000 \\ -0.433 & -0.750 & 0.000 \\ 0.433 & -0.750 & -0.433 \end{bmatrix} \begin{Bmatrix} U_{x1} \\ U_{y1} \\ U_{x3} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 15 \\ 15 \end{Bmatrix} \quad .$$

With global node (3) on rollers, the horizontal component of the reaction force at node (1) is also relieved. The vertical components of the reaction forces remain the same.