RESPONSE AND STRESS CALCULATIONS OF AN ELASTIC CONTINUUM CARRYING MULTIPLE MOVING OSCILLATORS

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ABSTRACT

The problem of calculating the dynamic response and stress of a one-dimensional distributed parameter system carrying multiple moving oscillators is examined. The number of oscillators, their speeds and their arrival times are arbitrary. A solution procedure is suggested that reduces the problem to the integration of a system of linear ordinary differential equations governing the time-dependent coefficients of the series expansion of the response in terms of the eigenfunctions of the continuous structure. The program implementation of the solution procedure is discussed and numerical results are presented to demonstrate the efficiency of this method and, in particular, its capability to accurately determine the jumps in the shear stress distributions.

KEYWORDS

Moving oscillator problem, distributed parameter system, response and stress calculations.

1. INTRODUCTION

The prediction of the dynamic response of a flexible structural system to moving loads and/or moving subsystems has been the subject of numerous investigations, most notably in the design of railroad tracks with high-speed trains, and bridges and elevated roadways with moving vehicles. These topics are extensively discussed in the review articles of Ting and Yener (1983), Diana and Cheli (1989), and Taheri et al. (1990), which give many additional references. Other important applications include band and circular saw blades (Mote, 1970), high speed machining (Chen and Wang, 1994), magnetic disk drives (Iwan and Moeller, 1976), and cables transporting materials/humans (Zhu and Mote, 1994).
The focus of this work is on the response of a flexible structure due to multiple moving oscillators. Two fundamental characteristics of these moving oscillator problems are reviewed: (1) no steady-state response exists, and (2) the dynamic interaction forces may be significant and response solutions can only be obtained by solving a set of coupled differential equations. In general, three types of problems have been reported in the literature. Dating back to the early work of Jeffcott (1929), the modeling of a vehicle traveling along a bridge as a moving force neglects the inertia of the moving subsystem and no dynamic interaction is considered (Fryba, 1972). When the inertia of the subsystem cannot be regarded as small, a moving mass model is employed (Sadiku and Leipholz, 1987). The inclusion of both the subsystem inertia and interaction forces in the moving oscillator problem is found in the study of vehicle-bridge interactions (Yang and Lin, 1995; Pesterev and Bergman, 1997; Yang et al., 2000). Most of the studies on the dynamic response of an elastic continuum to a stream of moving subsystems are concerned with the moving force problem (Li and Su, 1999; Sniady, 1989). To date, no systematic and efficient solution methodology exists for calculating the response of a continuum to a stream of arbitrary number of moving oscillators. The purpose of this paper is to develop a computation procedure for evaluating the dynamic response and stress of an elastic continuous structure due to an arbitrary number of oscillators traveling along the structure at arbitrary speeds.

2. PROBLEM STATEMENT

We consider the vibration of a spatially one-dimensional distributed parameter system due to multiple traveling linear oscillators (Fig. 1). In this paper, to simplify the discussion, the system is assumed to be conservative although the method discussed herein and the program implementation are applicable to proportionally damped continua and oscillators with damping as well. We first consider the case where the velocities of the oscillators are constant. Extension to the case of arbitrarily varying velocities is considered at the end of Section 3.

![Figure 1: Schematic of a distributed system carrying moving linear oscillators.](image)

Let the oscillators arrive at the left end of the continuum at \( t_i = 0, t_2, \ldots, t_\ell \), where \( \ell \) is the total number of oscillators. For no external forces, the governing equations are

\[
\rho \ddot{w}(x,t) + Kw(x,t) = -\sum_{i=1}^{\ell} \left( m_i g + f_i(t) \right) \left[ h(t-t_i) - h(t-(t_i + L/v_i)) \right] \delta(x-v_i(t-t_i)), \quad (1)
\]

\[
m_i \ddot{z}_i = f_i(t), \quad (2)
\]

where \( w(x,t) \) is the transverse displacement of the continuum; \( x \in [0,L] \), \( \rho \) and \( K \) are spatial differential positive definite operators representing the inertia and stiffness of the system; \( \delta(x) \) is the
Dirac-delta function; \( h(x) \) is the Heaviside unit-step function; \( m_i \), \( z_i(t) \), and \( v_i, i=1,\ldots,\ell \), are masses, relative displacements, and speeds of the oscillators, respectively; \( f_i(t) \) are interaction forces between the continuum and the oscillators, which are given by

\[
f_i(t) = k_i \left( w(v_i(t-t_i),t) - z_i(t) \right),
\]

and \( k_i \) is the stiffness of the spring connecting the mass \( m_i \) to the distributed system. We further assume that the distributed system has no rigid-body modes, and the boundary conditions are homogeneous and initial conditions are zero.

### 3. SOLUTION PROCEDURE

It is well known that the solution to equation (1) can be written in the form (e.g., Yang, 1996)

\[
w(x,t) = -\int_{-\infty}^{t} \int_{0}^{L} g(x,\eta,t-\tau) \sum_{i=1}^{\ell} \left( m_i g + f_i(\tau) \right) \left[ h(\tau-t_i) - h(\tau-(t_i+L/v_i)) \right] \delta(\eta-v_i(\tau-t_i)) d\eta,
\]

where \( g(x,\eta,t) \) is the dynamic Green’s function of the distributed system given by

\[
g(x,\eta,t) = \sum_{n=1}^{\infty} \frac{\varphi_n(x) \varphi_n(\eta)}{\omega_n} \sin(\omega_n t).
\]

Substituting (5) into (4) with a \( N \)-term expansion and performing integration gives

\[
w(x,t) = -\sum_{n=1}^{N} \varphi_n(x) q_n(t),
\]

where the \( q_n \)'s satisfy

\[
\ddot{q}_n + \omega_n^2 q_n = \sum_{i=1}^{\ell} F_{n,i}(t), \quad n = 1,\ldots,N.
\]

and,

\[
F_{n,i}(t) = \varphi_n(v_i(t-t_i)) \left( m_i g + f_i(t) \right) \left[ h(t-t_i) - h(t-(t_i+L/v_i)) \right] .
\]

The right-hand sides of equations (7) depend on the elastic forces \( f_i(t) \) that couple these equations and equation (2). Substituting (6) into (3), we obtain \( f_i(t) \) in terms of \( z_i(t) \) and \( q_n(t) \) as

\[
f_i(t) = -k_i \left( \sum_{n=1}^{N} \varphi_n(v_i(t-t_i)) q_n(t) + z_i(t) \right), \quad t_i \leq t \leq t_i + L/v_i.
\]

Simultaneous solution of equations (2) and (7) and the use of equations (6), (8), and (9) give the response of the continuum and the oscillators.

In general, there are two different approaches to implement the solution procedure described above. In the first approach, the number of equations (2) to be solved simultaneously in a particular time interval
depends on the interval being considered and may vary between 0 and $\ell$. The second approach solves the fixed number $N + \ell$ equations (2), (7) in any time interval, with the solutions to some equations being trivial. While the former approach may reduce the amount of computations, the latter results in simpler programming and more systematic notations of the equations. This is achieved by introducing the so-called extended eigenfunctions of the elastic structure. Define the eigenfunctions $\varphi_n(x)$, which are originally defined for $0 \leq x \leq L$, for all real values of $x$ by assuming that

$$\varphi_n(x) = 0 \quad \text{for} \quad x < 0 \quad \text{and} \quad x > L.$$ (10)

This definition makes (2) valid for all $t \in [0, \infty)$ and avoids the necessity to determine how many terms are to be used on the right-hand sides of (7) at a particular time $t$. Then, $F_{n,i}$ takes the form

$$F_{n,i}(t) = \varphi_n(v_i(t - t_i))(m_ig + f_i(t)),$$ (11)

i.e., we do not need to use the Heaviside functions. The functions $f_i(t)$ now become definite for all values of $t$, and, for $t < t_i$ or $t > t_i + L/v_i$, the right-hand sides of equations (9) take the form

$$f_i(t) = -k_i z_i(t).$$ (12)

Thus, we arrive at the following coupled set of $N + \ell$ ODEs valid for all $t > 0$:

$$\ddot{q}_n = -\omega_n^2 q_n + \sum_{i=1}^{\ell} \varphi_n(v_i(t - t_i))(m_ig + f_i(t)), \quad n = 1, \ldots, N, \quad q_n(0) = \dot{q}_n(0) = 0,$$ (13)

$$\dot{z}_i = \frac{1}{m_i} f_i(t), \quad i = 1, \ldots, \ell,$$ (14)

where $f_i(t)$ are defined by the unique equation

$$f_i(t) = -k_i \left( \sum_{n=1}^{N} \varphi_n(v_i(t - t_i))q_n(t) + z_i(t) \right), \quad i = 1, \ldots, \ell, \quad t \in [0, \infty).$$ (15)

Note that equations (13), (14) are also valid if there exist some time intervals during which no oscillators traverse the continuum. Equations (13), (14) can be rewritten in a standard way as a set of linear first-order differential equations and can be integrated by using any conventional technique such as the Runge-Kutta method. The above-described solution procedure was implemented in MATLAB and the ode23 function was used to integrate the set of first-order equations. The response of the distributed parameter system is then calculated by (6). In the beam case, series (6) converges rapidly such that a few terms are sufficient to obtain very good approximation of the response, which makes the method discussed very efficient. The time required to calculate the response depends on the complexity of the model under consideration and on the length of the time interval of interest. For the examples in Section 5, the calculation of the response took less than one minute on a standard PC.

Now consider the case of arbitrarily varying speeds of the moving oscillators. Introduce the notation $\zeta_i(t) = v_i(t - t_i)$, which denotes the position of the $i$th oscillator on the continuum at a given $t$. Then, the arguments $v_i(t - t_i)$ of the functions $\varphi_n(x)$ on the right-hand sides of equations (13) and (15) can be replaced by the functions $\zeta_i(t)$. Since we did not use the explicit dependence of the functions $\zeta_i(t)$
on time in all the intermediate calculations, it is evident that the equations remain valid if the \( \zeta(t) \) are arbitrary non-decreasing functions of time. For any particular problem, \( \zeta(t) \) can easily be evaluated given the laws of kinematics of the oscillators’ motion.

4. CALCULATION OF THE DERIVATIVES OF THE RESPONSE

Once \( q_n \)'s are obtained, the response of the continuum and all its derivatives can be evaluated. For instance, the shear force can be expanded in terms of the derivatives of the eigenfunctions as

\[
EI w_{xxx}(x,t) = -EI \sum_{n=1}^{N} \varphi_n(x) q_n(t).
\]

It is well known, however, that the convergence of the series obtained by differentiating (6) becomes worse for higher order of derivatives, and a large number of terms is required in the series to achieve a desired accuracy. Moreover, because of the singularities on the right-hand side of (1), the shear force has jumps at the points of the oscillator attachments, and expansion (16) fails to adequately approximate the shear force distribution in the moving oscillator problem. To overcome this difficulty, an improved series expansion has been suggested in Pesterev et al. (1999), which is based on the use of a quasi-static solution defined in terms of the static Green's function,

\[
EI w_{xxx}(x,t) = -EI \sum_{n=1}^{N} \varphi_n(x) q_n(t) - \sum_{i=1}^{L} F_i(t) EI \left( G_{xxx}(x,v_i(t-t_i)) - \sum_{n=1}^{N} \frac{\varphi_n(x) \varphi_n(v_i(t-t_i))}{\omega_n^2} \right),
\]

where \( G(x,\xi) \) is the static Green's function of the distributed system and \( F_i(t) = m_i g + f_i(t) \) is the force acting on the beam from the \( i \)th oscillator. The function \( G(x,\xi) \) is the solution to the static equation \( KG(x,\xi) = \delta(x-\xi) \) satisfying the given boundary conditions and, for many structures (such as beams with different boundary conditions or a string) can be determined analytically. As can be seen, the jumps in the shear force at the points \( x_i(t) = v_i(t-t_i) \) are calculated exactly by virtue of the static Green’s function and are equal to \( F_i(t) \). Since the elastic forces \( f_i(t) \) are determined from the beam response rather than by higher derivatives, they can be accurately calculated by using the conventional series expansion (15). Thus, we arrive at the following easy-to-implement procedure for stress calculation: (i) calculate the coefficients \( q_n(t) \) of expansion (6) by using the solution procedure described in the previous section, (ii) calculate the elastic forces \( f_i(t) \) by (15), and (iii) substitute \( f_i(t) \) into (17) to accurately calculate the shear force distribution.

5. NUMERICAL ILLUSTRATIONS

The basic aim of our numerical experiments was to examine the convergence of the expansions (6), (16), and (17) and to understand any new physical phenomena when more than one oscillator is included in the model. We considered the case where two equal oscillators (\( m_i = m_2 = m \) and \( k_1 = k_2 = k \)) traverse a simply supported beam with the same speed \( v_1 = v_2 = v \). Moreover, in the numerical experiments, only \( v \) and the arrival time interval \( \Delta T = t_2 - t_1 \) were allowed to vary. The beam parameters were the same as those employed in Sadiku and Leipholz (1987) and in Pesterev and Bergman (1997): \( L = 6 \text{ m} \), \( EI / \rho = 275.4408 \text{ m}^4 / \text{s}^2 \), \( m / \rho L = 0.2 \). The spring stiffness \( k = 2000 \text{ N/m} \) was sufficiently large to emulate a moving mass problem.
We first examined the relationship between the maximum response $\max, \max |w(x, t)|$ of the beam and $\Delta T$. This relationship is of practical importance: the values of $\Delta T$ for which the maximum response takes its minimal value may be viewed as optimal, whereas those corresponding to large amplitudes of the maximum response should be avoided. Our results show that, for a given speed, the largest of the maximum response of the beam always occurs when $\Delta T$ is zero (i.e., they travel together). As $\Delta T$ increases, the maximum response reaches its maximal and minimal values for certain $\Delta T$, which depend on both the velocity of the oscillators and the fundamental frequency of the beam.

Figure 2 illustrates the results for three cases: $v = 4, 6, 12$ m/s. Each curve in the plots depicts the displacement of the beam at the instant when the maximum response takes place for the given velocity and $\Delta T$. The solid, dashed, and dotted curves correspond to $\Delta T = 0, 1,$ and 1.5 seconds, respectively.

Figure 2: Maximum deflections of the beam due to two oscillators traveling at (a) 4 m/s; (b) 6 m/s; (c) 12 m/s. $\Delta T = 0$ (solid curves), $\Delta T = 1$ s (dashed curves), $\Delta T = 1.5$ s (dotted curves).

The values of time at which the beam displacements were calculated are shown in the plots. Note also that, in all cases depicted, the maximum response for $\Delta T = 1$ s (dashed curves) was due to only the first oscillator; i.e., the second oscillator does not increase the level of the vibration. It can be seen from the comparison of the solid and dashed curves in all plots that the magnitude of the maximum response for the $\Delta T = 0$ case is more than twice the magnitude of the maximum response due to one
moving oscillator. This phenomenon can be explained by the dynamic effect of the mass of the oscillators and by noticing that the mass of the beam is finite and is of the same order of magnitude as \( m \). If we did not consider the interaction between the masses and the beam (i.e., considered the moving force problem), the response would be exactly twice as much as that due to one weight.

In the above results, a five-term expansion was used to calculate the response. In fact, the convergence of series (6) is so good that even three terms give almost the same results. Figure 3 compares the convergence of the conventional series (16) and improved series (17) for the shear force in the two-oscillator example with \( v = 4 \) m/s, \( \Delta T = 1 \) s and spring stiffness \( k = 80 \) N/m. It demonstrates explicitly that the use of the improved series for the shear force calculation requires about the same number of terms as the conventional series for calculation of the responses, whereas the latter fails to adequately approximate the shear force even if the number of the series terms is large. Moreover, the improved series gives accurate values of the jumps in the shear force. The solid curve in Fig. 3 displays the exact shear force distribution at \( t = 1.25 \) s obtained through the use of ten terms of the expansion (17), the dashed curve shows the three-term approximation to the solution by means of the improved series (beginning with \( N = 4 \), all approximations coincide), and the dotted curve represents the ten-term approximation by the conventional series (16). The difference in the convergence is easily seen.

![Figure 3: Shear force distribution at \( t = 1.25 \) s: exact solution (solid curve) and its approximations obtained by using 3 terms of the improved series (17) (dashed curve) and 10 terms of the conventional series (16) (dotted curve) for two oscillators moving at 4 m/s.](image)

6. CONCLUSIONS

The response of a distributed parameter system due to an arbitrary number of oscillators traversing it has been obtained as a series expansion in terms of the eigenfunctions of the continuum. The differential equations governing the time-dependent coefficients of the expansion have been derived. The method presented is not specific to a particular conservative distributed parameter system and can be applied to either a string or a beam with arbitrary boundary conditions. Moreover, the oscillators are allowed to move with different and arbitrarily varying speeds.

The modeling of the oscillator arrivals and departures is simplified by defining the extended eigenfunctions, which in turn makes the program implementation much easier. Numerical results for a two-oscillator system are shown to demonstrate the dependence of the response on the arrival time interval between the oscillators and the convergence of the improved series (17) for the shear force,
which is shown to provide a high accuracy of calculation by means of a few terms of the expansion. Investigation of the dynamics of the multiple oscillators problem is underway.

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