Read 5.3

HW: Due next wed

Rewrite prog 4.1 so you don't use any generic complex operations.

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\[ a(n) \cos \omega t + b(n) \sin \omega t \]

Computation of the Discrete Fourier Transform.

The Discrete Fourier Transform plays an important role in the analysis, design and implementation of discrete-time signal processing algorithms and systems. It is thus very important to be able to compute DFT efficiently.
The efficient of computing DFT are collectively called Fast Fourier Transform Algorithms.

Recall that if \( x(n) \) and \( \tilde{x}(k) \) are DFT pairs \( \Rightarrow \)

\[
\tilde{x}(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}nk}
\]

to compute all terms of

the DFT we need to carry out \( N^2 \) complex multiplies and \( N(N-1) \) complex additions.

Let us represent \( x(n) \) in terms of a polynomial

\[
f(z) = x(n-1)z^{n-1} + x(n-2)z^{n-2} + \cdots + x(1)z + x(0)
\]
\[ \bar{x}(-k) = \frac{f(z)}{z = w^{-k}} = f( w^{-k} ) \]

\[ \bar{x}(-k) = \sum_{n=0}^{N-1} c_n \bar{z}^n \]

\[ \bar{z} = w^n \]

\[ \bar{x}(-k) = f( \bar{z} ) = x(3) \bar{z}^3 + x(2) \bar{z}^2 + x(1) \bar{z} + x(0) \]

\[ = f( (x(3) \bar{z} + x(2)) \bar{z} + x(1)) \bar{z} + x(0) \]

**Horner's rule**

In general, \( f(z) = \)

\[ (((x(N-1) \bar{z} + x(N-2)) \bar{z} + x(N-3)) \bar{z} + \ldots + x(1)) \bar{z} + x(0) \]

\[ f(z) \bigg|_{z = w^{-k}} \]

\[ \bar{x}(-k) = (((x(N-1) w^{-k} + x(N-2)) w^{-k} + \ldots + x(1)) w^{-k} + x(0) \]
Note that we only have \( N \) multiply-add operations per \( x(k) \) or \( N^2 \) MAC operations for all \( x(k) \) and require the storage of \( N \) complex exponentials.

Now suppose that we are interested in computing only a small subset of the DFT coefficients. An example of this is finding a known sinusoid in noise.

An algorithm that is very efficient for computing a subset of the DFT coefficients is the Goertzel Algorithm.
Note that \[ W_N^{KN} = 1 = e^{j \frac{2\pi k N}{N}} \]

Also recall that

\[ X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \]

\[ X(k) = \sum_{n=0}^{N-1} k(n) W_N^{k(n)} \]

Let us define a sequence

\[ y_k(m) = \sum_{n=-\infty}^{\infty} x(n) W_N^{k(m-n)} u(m-n) \]

\[ \sum_{n=0}^{\infty} x(n) W_N^{k(m-n)} \]

\[ \tilde{X}(k) = \left. y_k(m) \right|_{m=N} \]

Since \( x(n) \) is zero for \( n \leq 0 \) and \( n > N-1 \)
Consequently, \( y_k(m) \) can be viewed as the response of a system with impulse response \( w_n^{-k} u(n) \) to a finite-length input \( x(n) \). \( \delta(k) \) is the value of the output when \( m = N \). A system with such impulse response is shown below.

\[
\begin{align*}
    y(-1) &= 0 \\
    y(0) &= x(0) \\
    y(1) &= x(1) + w_n^{-1} x(0) \\
    y(2) &= x(2) + x(1) w_n^{-2} + x(0) w_n^{-1} \\
    \vdots \\
    y(n-1) &= x(n-1) + x(n-2) w_n^{-n} + \cdots + x(0) w_n^{-1} \\
    y(n) &= x(n) \\
    y(n+1) &= x(n+1) + \cdots + x(0) w_n^{-n-1}
\end{align*}
\]
Let $D y_k(m) = y_k(m-1)$

$y_k(m) = x(m) + D y_k(m-1) w_n^{-k}$

$= x(m) + D y_k(m-a) w_n^{-k}$

$\frac{y_k(m)}{x(m)} = \frac{1}{1-w_n^{-k}D}$

$\frac{1-w_n^{-k}D}{(1-w_n^{-k}) (1-w_n^{+k}D)}$

$= \frac{1-w_n^{-k}D}{1-(w_n^{-k}+w_n^{k})D+D^2}$

$= \frac{1-w_n^{k}D}{1-2\cos\left(\frac{2\pi k}{N}\right)D+D^2}$
The computation of $X(k)$ now requires one complex multiply (at $m = N$), since we only need to multiply by $W_N$ when $m = N$, and we need to carry out no real multiplies by $2 \cos \left( \frac{2\pi k}{N} \right)$ and we need $2N+1$ additions.