Read 7.5
HW:
7.1, 7.3, 7.21, 7.22
7.23

There are a number of sinusoids embedded in noise (rand.dat) on class web site. Find the frequency of these sinusoids. Hand in your program which computes the sinusoids as well as plots indicating or showing the signals. The sinusoids have frequencies less than 1000 Hz & thus are sampled at 1200 Hz
As discussed before, for a LTI System to be BIBO stable, \[ \sum_{n=-\infty}^{\infty} |h(n)| < \infty \]

Systems having a finite duration impulse response are always stable. Since the sum of a finite number of finite valued terms is always finite:

\[ y(n) = b_0 x(n) + b_1 x(n-1) + \ldots + b_q x(n-q) \]
We thus need to consider the infinite duration component. This component is generated by the denominator poly.

Transfer function.

Let us consider a simple (All pole) transfer function

\[ H(z) = \frac{1}{1 + \sum_{k=1}^{M} a_k z^{-k}} \]

The Fundamental theorem of Algebra applied to poly.
indicated that we can factor the denominator into linear terms, so
\[ H(z) = \sum_{k=1}^{M} \frac{c_k}{1 - d_k z^{-1}} \]

\[ \frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} \]

\[ c_k, d_k \in \mathbb{C} \]

Since the Z-Transform is linear, we know that
\[ \mathcal{F}^{-1} \{ H(z) \} = h(n) = \sum_{k=1}^{M} h_k(m) \]

However, we have seen (last class) that
\[ h_k(n) = c_k \cdot d_k^n \cdot u(n) \]

For the system to be stable, each \( h_k(n) \) must satisfy

\[ \sum_{n=-\infty}^{\infty} |h_k(n)| < \infty \Rightarrow \]

\[ \sum_{n=-\infty}^{\infty} |c_k d_k^n| = |c_k| \sum_{n=0}^{\infty} |d_k|^n \]

\[ \Rightarrow |d_k| < 1 \]

This implies that the poles must reside inside the unit circle.
Relationship between the DTFT & the Z-transform

The complex variable $Z$ can be expressed as $Z = r e^{j\omega}$, which is in vector form.

We then can write the Z-transform

$$H(re^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n) r^{-n} e^{-j\omega n}$$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n) e^{-j\omega n}$$

Thus $H(r e^{j\omega}) = H(z) \bigg|_{z = e^{j\omega}}$

If $r = 1$, then $H(re^{j\omega}) = H(e^{j\omega})$
The Inverse Z-Transform.

Given \( H(z) \) and a ROC
what is \( h(n) \)?

1. Inverse is computed on a polynomial.

   \[
   \text{ex. let } H(z) = a z + b + c z^{-1}
   \]

   \[
   h(-1) = a, \quad h(0) = b, \quad h(1) = c
   \]

2. Long Division method.

   \[
   \text{ex. } H(z) = \frac{1}{1 - a z^{-1}} = \frac{z}{z - a}
   \]

   \[\text{ROC } |z| > a\]
\[ \frac{z}{z-a} = 1 + a z^{-1} + a^2 z^{-2} + \cdots \]
\[ = \sum_{n=0}^{\infty} (az^{-1})^n = \sum_{n=0}^{\infty} a^n z^{-n} \]
\[ \implies h(n) = \begin{cases} a^n & \text{for } n \geq 0 \\ 0 & \text{otherwise} \end{cases} \]
\[ = a^n u(n) \]

**Example:**

\[ H(z) = \frac{1-b^2 z^{-2}}{1+a^2 z^{-2}} \quad |z| > a \]

\[ H(z) = 1 - (a^2 + b^2) z^{-2} + a^2 (a^2 + b^2) z^{-4} \]

\[ h(0) = 1, \quad h(1) = 0, \]
\[ h(2) = -(a^2 + b^2), \quad h(3) = 0 \]
\[ h(4) = a^2 (a^2 + b^2) \]
Long division can be applied to any rational function. However, we usually need to guess the values (coefficients) of \( h(n) \).

3. Power series (Geometric series) and partial fraction expansion.

\[
H(z) = \frac{1}{1 - a z^{-2}} \quad |z| > |a|^{1/2}
\]

\[
H(z) = \sum_{n=0}^{\infty} (az^{-2})^n = \sum_{n=0}^{\infty} a^n z^{-2n}
\]
\[ h(n) = \begin{cases} a^{n/2} & \text{for } n \geq 0 \mod 2 \\ 0 & \text{otherwise} \end{cases} \]

Example:

\[
\frac{1 - b^2 z^{-2}}{1 + a^2 z^{-2}} = H(z)
\]

\[
H(z) = \frac{1}{1 + a^2 z^{-2}} - \frac{b^2 z^{-2}}{1 + a^2 z^{-2}}
\]

\[
\sum_{n=0}^{\infty} (-a^2 z^{-2})^n = -b^2 z^{-2} \sum_{n=0}^{\infty} (-a^2 z^{-2})^n
\]

\[
= 1 - (a^2 + b^2) z^{-2} + a^2 (a^2 + b^2) z^{-4} - a^4 (a^2 + b^2) z^{-6} + \cdots
\]
\[ H(z) = \frac{r \sin(w_0) z^{-1}}{1 - 2r \cos(w_0) z^{-1} + r^2 z^{-2}} \]

Factor the denominator

\[ H(z) = \frac{r \sin(w_0) z^{-1}}{(1 - r e^{jw_0} z^{-1})(1 - r e^{-jw_0} z^{-1})} \]

Now use partial fraction expansion to write:

\[ H(z) = \frac{A}{1 - r e^{jw_0} z^{-1}} + \frac{B}{1 - r e^{-jw_0} z^{-1}} \]

\[ = \frac{A - A r e^{-jw_0} z^{-1} + B - B r e^{jw_0} z^{-1}}{D(z)} \]

\[ r \sin(w_0) z^{-1} = A - A r e^{-jw_0} z^{-1} + B - B r e^{jw_0} z^{-1} \]

\[ A = -B, \quad A = \frac{1}{2j} \]
\[ H(z) = \frac{1}{2j} \left( \frac{1}{1 - re^{j\omega_0}z^{-1}} - \frac{1}{1 - re^{-j\omega_0}z^{-1}} \right) \]

\[ = \frac{1}{2j} \sum_{n=0}^{\infty} \left( re^{j\omega_0}z^{-1} \right)^n - \frac{1}{2j} \sum_{n=0}^{\infty} \left( re^{-j\omega_0}z^{-1} \right)^n \]

\[ = \frac{1}{2j} \sum_{n=0}^{\infty} r^n (e^{j\omega_0} - e^{-j\omega_0}) (z^{-1})^n \]

\[ = \sum_{n=0}^{\infty} \left( r^n \sin \omega_0 \right) z^{-n} \]

\[ h(n) = r^n \sin \omega_0 \delta(n) \]