Please Read Chapter 8.

$\mathcal{Z}$-Transform

$$H(z) = \sum_{n=0}^{\infty} h(n)z^{-n}$$

Residue Method.

From complex analysis we know that
\[ h(n) = \frac{1}{2\pi j} \oint_C (H(z)z^{n-1})dz \]

(proof is in the book)

C is a closed contour in counter clockwise direction that contains the ROC of \( H(z) \).

Let \( H(z)z^{n-1} \) have \( N \) poles \( P_1, P_2, \ldots, P_N \), then

\[ h(n) = \sum_{i=1}^{N} \text{Res} \left[ \frac{H(z)z^{n-1}}{z - P_i} \right] \]

Let \( H(z)z^{n-1} \) be a rational function at \( z \) with an \( m^{th} \)
order pole at $z = z_0$.

We then can write $H(z) z^{n-1}$ as

$$H(z) z^{n-1} = \frac{P(z)}{(z-z_0)^m}$$

Note that $P(z)$ has no pole at $z = z_0$.

The residue of $H(z) z^{n-1}$ at $z = z_0$ is

$$\text{Res} \left[ H(z) z^{n-1} \right]_{z = z_0} = -$$

$$\frac{1}{(m-1)!} \left. \frac{d^{m-1} P(z)}{d z^{m-1}} \right|_{z = z_0}$$
For the simple case of $n > 0$

$$\text{Res} \left[ H(z) z^{n-1} \bigg|_{z=\zeta_0} \right] = P(\zeta_0)$$

**Example:**

$$H(z) = \frac{1}{1-az^{-1}} \quad \text{Roc } |z| > |a|$$

$$h(n) = ?$$

$$H(z) z^{n-1} = \frac{z^n}{z-a}$$

$$P(z) = H(z) z^{n-1} (z-a) = z^n$$

$$= P(a) = a^n \nu(n)$$

for $n > 0$
\[ H(z) = \frac{1 - b^2 z^{-2}}{1 + a^2 z^{-2}} \quad \text{Roc} \quad |z| > |a| \]

\[ H(z) z^{n-1} = \frac{(z^2 - b^2) z^{n-1}}{(z + ja)(z - ja)} \]

For \( n = 0 \) there are 3 poles:

\[ h(0) = -\frac{b^2}{a^2} + \frac{a^2 + b^2}{2a^2} + \frac{a^2 + b^2}{2a^2} \]

For \( n \geq 1 \) \( H(z) z^{n-1} \) has 2 poles:

\[ h(n) = \frac{(z^2 - b^2) z^{n-1}}{z + ja} \bigg|_{z = ja} + \frac{(z^2 - b^2) z^{n-1}}{z - ja} \bigg|_{z = -ja} \]
\[ \frac{a^2 + b^2}{2ja} \left[ (-ja)^{n-1} - (ja)^{n-1} \right] \]

\[ = (-1)^{n/2} a^{n-2} (a^2 + b^2) \]

Digital Filter Design

Filtering in general is about manipulating the spectral contents of signals.

Analog Filters are described in S-domain

\[ H(s) = \frac{b_0 s^q + b_1 s^{q-1} + \ldots + b_q}{s^p + a_1 s^{p-1} + \ldots + a_p} \]

\[ q \leq p \]
Digital filters are described in the $z$-domain:

\[ H(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_N z^{-N}}{1 - a_1 z^{-1} - \cdots - a_P z^{-P}} \]

$H(z)$

Filters for which the degree of the denominator is larger or equal to one are called IIR (Infinite Impulse Response).

This is because $H(z)$ can
be factored such that it contains term of the type

\[ \sum_{k=1}^{P} \frac{A_k}{1-\lambda_k z^{-1}} = \]

\( h(n) \) will contain terms of the type

\[ \sum_{k=1}^{P} A_k \lambda_k^n u(n) \]

Otherwise the filter is called Finite Impulse Response (FIR)

this the case since the response due to an impulse vanishes after at most \((P+1)\) terms
Remember that for this case $H(z)$ has no poles.

\[ H(z) = \frac{Y(z)}{X(z)} = b_0 + b_1 z^{-1} + \cdots + b_8 z^{-8} \]

\[ y(z) = b_0 \bar{X}(z) + b_1 z^{-1} \bar{X}(z) + \cdots + b_8 z^{-8} \bar{X}(z) \]

\[ y(n) = b_0 x(n) + b_1 x(n-1) + \cdots + b_8 x(n-8) \]

$x(n) = \delta(n)$, then $y(n) = h(n)$

\[ h(0) = b_0 \]
\[ h(1) = b_1 \]
\[ h(2) = b_2 \]
\[ h(8+1) = 0 \]
Filter Types:

Let us first give a short definition of different filter types:

1. Low Pass Filter:

Are designed to pass low frequencies up to some frequency, $W_p$
2- High pass filters:
   Are designed to pass high frequencies from \( \tilde{\omega}_p \) up to \( \pi \).

3- Band pass filters:
   Are designed to pass a certain range of freq.
4. Band Stop Filters:

Are designed to block a range of frequencies.
Of course filters are not ideal. The response of filters are divided into 3 sections: Pass band, Transition band, Stop band.

Low pass filter Spec:

Pass band ripple

Stop band ripple

Transition band width
The ripples are typically expressed in dB as:

\[ \text{in dB} = 20 \log \left( \frac{1}{10} \right) \text{ (Peak-to-Peak ripple)} \]

High pass filters

Stop \( \delta_s \) = 50 dB

Pass band ripple < 0.2 dB

\( \tilde{\omega}_s = 0.2 \pi \)

\( \tilde{\omega}_p = 0.25 \pi \)
So far we have only discussed the magnitude (Amplitude) response of digital filters but equally important is the phase response of digital filters.
In many cases we would like to design filters which have linear phase response. That is the phase is linearly proportional to the frequency.

At times we would like to design filters that implement constant phase shift.
Such as $-90^\circ$ (Hilbert Transformer)

\[ H(e^{j\omega}) = \begin{cases} 1 & \text{for } |\omega| < \omega_p \\ 0 & \text{otherwise} \end{cases} \]

\[ h(n) = \frac{1}{\pi} \int_{-\omega_p}^{\omega_p} e^{j\omega n} d\omega \]