Design a FIR bandpass filter with frequency response

$$\frac{\pi}{40} \leq \omega \leq \frac{3\pi}{8},$$

such that we get at least 60 dB attenuation in the stopband and a width of the transition

$$\leq \frac{\pi}{20}.$$ 

Plot the magnitude response of the filter, and the spectrum of the speech signal filtered with this filter.
Problem 10.5

Read sections 10.3 and 10.6

Properties of Butterworth Filters

1. Magnitude response is a monotonically decreasing function of \( w \).

2. The maximum gain occurs at \( w = 0 \) and \( \left| H(0) \right| = 1 \)

3. We have \( \left| H(\omega_0) \right| = \sqrt{2} \) which is the 3dB point.
The asymptotic attenuation at high frequencies is \(20N \text{ dB/decade}\) or \(6N \text{ dB/octave}\).

5. The square magnitude response

\[
\frac{1}{\frac{d}{dw} \left| H(w) \right|^2} \left| \frac{1}{w_0} \right| = 0
\]

\(6 \text{ dB/oct}\)

\[
\frac{1}{2} \leq k \leq 2N - 1
\]

The frequency response is nearly flat at low frequencies. That is why the Butterworth Filters are called maximally flat filters.
The transfer function of Butterworth filters can be obtained in the following way:

\[
H(s)H(-s) = \frac{1}{1 - (\frac{s}{j\omega_0})^{2N}} = \frac{1}{1 - (-1)^N (\frac{s}{\omega_0})^{2N}} \left(\frac{j^2}{-1}\right)^N
\]

which has \(2N\) poles.

The poles are \(s_k = \omega_0 \exp\left\{j(\pi(k+1)/2)\right\}\) for

\[0 \leq k \leq 2N-1\]

The poles on the left half plane belong to \(H(s)\). There are the poles corresponding to \(0 \leq k \leq N-1\).
Butterworth filters have no zeros, and we can write

\[ H(s) = \sum_{k=0}^{N-1} \frac{-s^k}{s - s^k} \]

\[ H(s) = \frac{1}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + 1} \]

where the coefficients are normalized such that \( H(0) = 1 \).

The design procedure for a Butterworth filter is as follows:
We first compute the discrimination factor \( d \) and the selectivity factor \( k \) from the given design parameters \((1 - \delta P), \delta S\)

\[
wp \text{ and } ws \\
\frac{wp}{ws}
\]

\[
d = \frac{(1 - \delta P)^{-2} - 1}{\delta S^{-2} - 1}
\]

we then have

\[
\frac{1}{1 + \left(\frac{wp}{ws}\right)^{2n}} \geq (1 - \delta P)^2
\]

\[
\frac{1}{1 + \left(\frac{ws}{ws_0}\right)^{2n}} \leq \delta S^2
\]
\[
\left( \frac{\omega_s}{\omega_p} \right)^{2N} \geq \frac{\delta_s^{-2} - 1}{(1 - \delta_p)^{-2} - 1}
\]

\[
\left( \frac{1}{N} \right)^{2N} \geq \left( \frac{1}{N_d} \right)^2 = \frac{\log_e (\frac{1}{N_d})}{\log_e (\frac{1}{N_k})} \geq N \geq \sqrt{\frac{\log_e (\frac{1}{N_d})}{\log_e (\frac{1}{N_k})}}
\]

and

\[
\omega_p \left[ (1 - \delta_p)^{-2} - 1 \right] \leq \omega_0 \leq \omega_s \left[ \delta_s^{-2} - 1 \right]^{-\frac{1}{2N}}
\]

\[
\left( \frac{1}{\sqrt{2}} \right)^2 = .5
\]

\[
|H(\omega)|^2
\]
Chebyshev Filters:

Note that $\cos(Nw)$ is a polynomial of degree $N$ in $\cos(w)$.

**Ex:**

$$\cos(2w) = 2\cos^2(w) - 1$$

$$\cos(3w) = 4\cos^3(w) - 3\cos(w)$$

$\cosh(Nw)$ is also a polynomial of degree $N$ in $\cosh(w)$. The Chebyshev polynomial of degree $N$ is defined as
\[ T_N(x) = \begin{cases} \cos(N\cos^{-1}(x)) & |x| \leq 1 \\ \cosh(N\cosh^{-1}(x)) & |x| > 1 \end{cases} \]

Chebyshev polynomials can be constructed by the following recursive formula:

\[ T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) \]

\[ T_0(x) = 1 \quad \text{and} \quad T_1(x) = x \]

Chebyshev polynomials have the following properties:

1. For \(|x| \leq 1\), \(|T_N(x)| \leq 1\) and \(T_n(x)\) oscillates between \(-1, 1\) a number of times proportional to \(N\).
2. $|x| > 1$, $T_N(x) ≥ 1$, and is monotonically increasing in $|x|$

3. Chebyshev poly of odd orders are odd functions of $x$, i.e., they only contain odd powers of $x$.

Chebyshev poly of even order are even functions of $x$, and only contain even powers of $x$.

4. $T_N(0) = ±1$ for $N$ even
   $0$ for $N$ odd
5. \[ |T_N(\pm 1)| = 1 \quad \forall N \]

The main property of the Chebyshev poly. is that, it oscillates in the range \(|x| \leq 1\) is is monotonic outside this range. This property is used to construct filters that are equi-ripple either in stopband or passband. Thus the magnitude oscillates a number of times between a MAX and a MIN value.
The equi-ripple property provides sharper transition between passband and stopband, and as a result, the order of the Chebyshev filter is smaller than that of the Butterworth filter for the same requirements. Chebyshev filters of the first kind. These filters are equi-ripple in the passband and monotonically...
decreasing in the stopband

\[ |H(w)|^2 = \frac{1}{1 + \sum T_n^2 \left( \frac{w}{\omega_0} \right)} \]

For \( N = 2 \):

\[ (1 - \delta_p)^2 \]

\[ SS \]

\[ wp \quad ws \]

For \( N = 3 \):

\[ (1 - \delta_p)^2 \]

\[ SS \]

\[ wp \quad ws \]

The properties of the Chebyshev filters are:
1. \[ 1 - \frac{1}{1 + \varepsilon^2} |H(\omega)|^2 \leq 1 \]

2. \[ 2 - |H(0)|^2 = \begin{cases} \frac{1}{1 + \varepsilon^2} & \text{N even} \\ \frac{1}{1 + \varepsilon^2} & \text{N odd} \end{cases} \]

3. For \( \omega > \omega_0 \), the response decreases monotonically and approaches \( 6N \text{dB/octave} \) or \( 20N \text{dB/decade} \). Similar to the Butterworth filters.

The poles of an \( N \)th order Chebyshev filter of the first kind are:
\[ S_k = -c_0 \sinh \left( \frac{1}{N} \sinh^{-1}(\frac{1}{2}) \right) \sin \left[ \frac{(2k+1)\pi}{2N} \right] \]

\[ + \frac{c_0}{2} \cosh \left( \frac{1}{N} \sinh^{-1}(\frac{1}{2}) \right) \cos \left[ \frac{(2k+1)\pi}{2N} \right] \]

\[ 0 \leq k \leq N-1 \]

The inverse hyperbolic functions can be computed using:

\[ \sinh^{-1}(x) = \log \left( x + \sqrt{x^2 + 1} \right) \]

\[ \cosh^{-1}(x) = \log \left( x + \sqrt{x^2 - 1} \right) \]