Read 11.1

Do 11.4 and 11.5

Low pass - to band pass

Transformation:

\( \frac{1}{\omega_c} \)

\[ (1 - \delta p)^2 \]

\[ \delta_p^2 \]

\[ \omega_p \]

\[ \omega_s \]
\[
S = \frac{\hat{S}^2 + \hat{w}_l \hat{w}_h}{\hat{S} (\hat{w}_h - \hat{w}_l)}, \quad w = \frac{\hat{w}^2 - \hat{w}_l \hat{w}_h}{\hat{w} (\hat{w}_h - \hat{w}_l)}
\]

\(\hat{w}_h > \hat{w}_l\)

Suppose that we want to design a BP filter with params: \(\tilde{\delta}_p, \tilde{\delta}_s_1, \tilde{\delta}_s_2\) and edge freqs \(\tilde{\omega}_{s_1}, \tilde{\omega}_p, \tilde{\omega}_p, \tilde{\omega}_{s_2}\)

Design BF based on our low pass prototype.

Let \(\delta_p = \tilde{\delta}_p\)
\[ s_s = \min \{ \tilde{s}_{s1}, \tilde{s}_{s2} \} \]

Let \( \tilde{w}_p = 1 \) and \( \tilde{w}_h = \tilde{w}_{p2} \)

Note that
\[
\frac{\tilde{w}_p^2 - \tilde{w}_{p1} \tilde{w}_{p2}}{\tilde{w}_{p1} (\tilde{w}_{p2} - \tilde{w}_{p1})} = -1 = \tilde{w}_p
\]

\[
\frac{\tilde{w}_{p2} - \tilde{w}_{p1} \tilde{w}_{p2}}{\tilde{w}_{p2} (\tilde{w}_{p2} - \tilde{w}_{p1})} = 1 = \tilde{w}_p
\]

\[
\tilde{w}_{s1} = \frac{\tilde{w}_{s1}^2 - \tilde{w}_{p1} \tilde{w}_{p2}}{\tilde{w}_{s1} (\tilde{w}_{p2} - \tilde{w}_{p1})}
\]

\[
\tilde{w}_{s2} = \frac{\tilde{w}_{s2}^2 - \tilde{w}_{p1} \tilde{w}_{p2}}{\tilde{w}_{s2} (\tilde{w}_{p2} - \tilde{w}_{p1})}
\]
$$w_5 = \min \left[ w_{s_1}, w_{s_2} \right]$$

BP Filter design procedure is as follows:

Given \( \tilde{\omega}_p, \tilde{\omega}_s, \tilde{\omega}_c, \tilde{\omega}_{s_2}, \tilde{\delta}_p, \tilde{\delta}_s, \tilde{\delta}_{s_2} \), choose \( \delta_p, \delta_s \) and \( \delta_{s_2} \), choose \( \delta_p, \delta_s \) and

\( w_6, w_7 \) according to above.

Let \( \omega_p = 1, \omega_{s_1} = \tilde{\omega}_p, 1 \) and \( \omega_{s_2} = \tilde{\omega}_p \). Compute \( w_{s_1} \) and \( w_{s_2} \) based on the above and choose \( w_5 = \min \left[ w_{s_1}, w_{s_2} \right] \).
Design the prototype lowpass filter and convert using:

\[
S = \frac{s^2 + \omega_n \omega_h}{s (\omega_h - \omega_L)} = \frac{s^2 + \tilde{\omega}_p \tilde{\omega}_L}{s + (\tilde{\omega}_L - \tilde{\omega}_p)}
\]

\[
H(s) = \frac{V_{RC}}{s + V_{RC}}
\]

Bilinear Transformation

\[s - \text{plane} \quad \leftrightarrow \quad z - \text{plane}\]
\[
\sinh(x) = \frac{1}{2} (e^x - e^{-x}) \quad x \in \mathbb{C}
\]
\[
\cosh(x) = \frac{1}{2} (e^x + e^{-x})
\]
\[
\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1}
\]
\[
\tilde{x} = 2x \Rightarrow
\]
\[
\tanh\left(\frac{\tilde{x}}{2}\right) = \frac{e^{\tilde{x}} - 1}{e^{\tilde{x}} + 1}
\quad x \in \mathbb{C}
\]
Bilinear transform is a way to convert analog filters to digital filters. This is accomplished by mapping the entire left half plane to the inside of the unit circle.

The Bilinear transform is:

\[
\begin{align*}
\tilde{\omega} & \rightarrow \frac{e^{\tilde{\omega}} - 1}{e^{\tilde{\omega}} + 1}, \quad \frac{z-1}{z+1} \quad \text{or} \\
\frac{S}{2} & \rightarrow \frac{z-1}{z+1} \quad \text{or} \\
S & \rightarrow \frac{2(z-1)}{(z+1)}; \quad S \rightarrow \frac{2}{\pi} \frac{(z-1)}{(z+1)}
\end{align*}
\]
ex: If \( H(s) = \frac{a}{s+a} \) then

\[
H(z) = \frac{\frac{a}{T}}{\left(\frac{z-1}{z+1}\right) + a} = \frac{1 + z^{-1}}{1 - \left(\frac{1 - 5aT}{1 + 5aT}\right) z^{-1}}
\]

Note that under the above transformation, \( S = \frac{z}{T} = \frac{z^{-1}}{z+1} \).

We get \( z = \frac{1 + ST/2}{1 - ST/2} \).

If we substitute \( S = 0 \) and compute the magnitude of the above.
\[ |z| = \sqrt{\frac{(1+\sigma T/2)^2 + (\omega T/2)^2}{(1-\sigma T/2)^2 + (\omega T/2)^2}} \]

It is clear that

\[ |z| < 1 \iff \sigma < 0 \]

At \( \sigma = 0 \) we get \( |z| = 1 \)

At \( \sigma = 0 \) we have

\[ z = e^{i\tilde{\omega}} = \frac{1 + i \omega T/2}{1 - i \omega T/2} \implies \]

\[ \tilde{\omega} = 2 \tan^{-1}(\omega T/2) \]

Therefore even though the transformation is one-to-one, the frogs are warped according to the above.
Therefore we will pre-warp the freq. prior to the design

\[ W_p = \frac{2}{T} \tan \left( \frac{\tilde{W}_p}{2} \right), \]

\[ W_s = \frac{2}{T} \tan \left( \frac{\tilde{W}_s}{2} \right) \]

prewarping is done before the analog filter design and the bilinear transformation is done after. We can then choose \( T = 1 \), or \( T = 2 \).

But we need to be sure that for do this for both pre-warping and bilinear transformation.
1. Convert the band edge freq.
   of the digital filter to the
   corresponding band edge freq.
   of the analog filter using
   the pre-warping formulas.
   (Tolerances remain the same)

2. Design the analog filter
   according to specs.

3. Transform $H(s)$ to $H(z)$
   using the bilinear transform.
Direct Method of Realizing

IIR Filters:

\[
H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_N z^{-N}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}
\]

\[
= \frac{B(z)}{A(z)}
\]

\[
H_1(z) = \frac{B(z)}{A(z)} \quad \text{and} \quad H_2(z) = \frac{1}{A(z)}
\]

\[
H(z) = \frac{Y(z)}{n(z)} = \frac{U(z)}{X(z)}
\]

\[
H_1(z) = \frac{1}{A(z)} = \frac{U(z)}{X(z)}
\]

\[
H_2(z) = B(z) = \frac{Y(z)}{U(z)}
\]
\[ H(z) = \frac{1}{A(z)} = \frac{U(z)}{X(z)} \]

\[ u(n) = -a_1 u(n-1) - a_2 u(n-2) - \ldots - a_N u(n-N) + x(n) \]

and \[ H(z) = \frac{1}{B(z)} = \frac{Y(z)}{U(z)} \]

\[ y(n) = b_0 u(n) + b_1 u(n-1) + \ldots + b_N u(n-N) \]