Random Signals

Recall a random variable, $x(u)$, is a functional relationship between a random event $u$ and a real no.

$x(u) \rightarrow f(x)$ — probability density function

Mean of $X$ or $E(X) = \int \infty_{-\infty} x f(x) dx$

Mean of $X^2$ or $E(X^2) = \int \infty_{-\infty} x^2 f(x) dx$

Variance $V(X) = E(X - E(X))^2$

$= E(X^2) - [E(X)]^2$
A random process is a function of two variables. An event \( \mu \) and time \( x(\mu, t) \) is an random variable \( (RV) \). In the strict sense, statistics are independent of time. Random processes are used to describe phenomena. The mean and second moment are independent of time.
Given any time \( t \)

\[
x(u, t) |_{t=\tau} = x(u, \tau)
\]

is an Random Variable (RV)

A Random process is said to be strict sense stationary if the overall statistics are independent of time.

A Random process is said to be wide sense stationary if the mean and second moment are independent of time.
A RANDOM PROCESS IS SAID TO BE WHITE IF $G(f)$ IS CONSTANT OVER ALL FREQUENCIES OF INTEREST.

\[
\text{PSD} \quad \rightarrow \quad G(f) \quad \text{white noise} \quad \rightarrow \quad N_0/2
\]

\[
\mathbb{P}_\text{av} = \int_{-\infty}^{\infty} G(f) df = \frac{N_0}{2} \int_{-\infty}^{\infty} df
\]

\[
= \infty.
\]

**Bandlimited Processes**

\[
G_{N_i}(f) \quad \overset{\text{white random process}}{\rightarrow} \quad \text{Ideal LPF} \quad \rightarrow \quad N_i(k, t) \quad \rightarrow \quad G_{N_i}(f) = G(f I | H(f))
\]

\[
H(f) = \begin{cases} 1 & \text{for } |f| \leq B \\ 0 & \text{otherwise} \end{cases}
\]
A random process is said to be white if $G_H(f)$ is constant over frequencies.

Random processes and linear systems:

\[
\begin{align*}
X(u, t) & \xrightarrow{G_x(f)} H(f) & Y(u, t) \\
& \xrightarrow{G_y(f)} &
\end{align*}
\]

Time invariant linear system:

Then,

\[
G_y(f) = G_x(f) \left| H(f) \right|^2
\]
IF A PROCESS IS WSS
Then THE POWER SPECTRAL DENSITY (PSD) IS
Given by
\[ G(f) = \int_{-\infty}^{\infty} R(\tau) e^{-j2\pi f \tau} d\tau \]
AND THE AVERAGE POWER IN THE PROCESS IS EQUAL TO
\[ P_{av} = \int_{-\infty}^{\infty} G(f) df \]
A process is WSS if its mean and autocovariance are constant for all \( \tau \), i.e.,
\[
\begin{align*}
x(x_1, t_1) &= \mu_0(t_1) \\
x(x_2, t_2) &= \mu_0(t_2) \\
x(x_3, t_3) &= \mu_0(t_3)
\end{align*}
\]

And the average power in the process is equal to
\[
P = \sigma^2
\]

And, the autocovariance is:
\[
x(x_1, t_1) = x(x_2, t_2) = x(x_3, t_3)
\]
Sample the process at two times \( t_j \neq t_k \). Form the RVs \( x(u, t_j) \neq x(u, t_k) \)

\[ \text{IF} \quad E(x(u, t_j)) = k \quad \text{II of Time any } t_j \]

AND

\[ E[x(u, t_j) x(u, t_k)] = R(\uparrow) \]

where \( \uparrow = t_k - t_j \)

\[ \text{IF} \quad R(t_j, t_k) = R(\uparrow) \]

\[ \text{THEN} \quad x(u, \uparrow) \quad \text{is WSS} \]
\[ x(\mu_0, t) \]

Solve the process at two times \( t \) and \( t + \Delta t \) from the R.V.s \( x(u, t_j) \).

\[ x(\mu_1, t) \]

IF \( \mathbb{E}[x(u, t_j) x(u, t_{j+1})] = \mathbf{R}(t) \)

THEN

\[ x(\mu_{w1}, t) \]

AUTO CORRELATION FUNCTION IS \( \mathbf{R}(t) \)

IF \( t = t' + \Delta t \)

\[ x(\mu_{1t}, t) \]

ARV
\[ G_+(f) \]

\[ N(0/2) \]

\[ -B \quad B \quad f \]

\[ \mathbb{P}_{AV} = \int G_{N_0}(f) \, df = \frac{N_0}{2} (2B) = N_0B \]

\[ G_{N_0}(f) \]

\[ \text{Ideal BPf} \]

\[ H(f) \]

\[ G_m(f) \]

\[ N(k,t) \]

\[ G_{N_0}(f) \]

\[ -f_m \quad -f_L \quad f_L \quad f_m \]

\[ G_{N_0}(f) = \frac{N_0}{2} \left[ u(f-f_L) - u(f-f_m) \right] + \text{Negative freq. term} \]
Gaussian Processes
Random

Now select a sample
of $H(u, t)$ \sim $N(t, k)$

$N(t, k)$ is a RV.

Can show that in a number of practical cases the PDF
of RV $N(t, k)$ \sim $N$

follows the Gaussian Probability Law.
WHERE $A$ and $B$ are CONSTANTS that depend on TEMPERATURE AND OTHER PHYSICAL CONSTANTS.

\[ G(f) = \frac{N}{N_c} \]

\[ f = 10^3 \text{ Hz} \]

For $f \ll 10^3 \text{ Hz}$, the graph shows:

\[ \text{WHITE NOISE} \]

\[ N_D = \text{Single Sided Noise Power Spectral Density} \]

\[ N_0 = kT \text{ Watts/Hz} \]
Thermal noise is a zero mean Gaussian random process with power spectral density:

\[ G(f) = A 1 f^1 \]

\[ \frac{N}{\sqrt{\frac{B}{f^1}} - 1} \]

Where \( A \) and \( B \) are constants that depend on temperature and other physical constants.

\( N_0 \) = Single Sided Noise Power Spectral Density

\[ N_0 = kT \text{ Watts/Hz} \]
Gaussian Processes
And Linear Systems

Thermal noise is a zero mean Gaussian random process.

\[ S(k,t) \rightarrow H(f) \rightarrow R(k,t) \]

Gaussian random process

**Example**

\[ \widetilde{S}(k,t) \]

is a zero mean white Gaussian noise process with PSD \( \frac{K_0}{2} \). 

\[ H(f) = \begin{cases} 1 & |f| \leq B \\ 0 & |f| > B \end{cases} \]

\( R(k,t) \) is a new zero mean band-limited white Gaussian noise process.

PDF - Gaussian

\( \sigma^2 = 2N_0B \)

\( M_R = 0 \)
\[ f(n) = \frac{E + p \left( \frac{(n - M_n)^2}{2\sigma^2} \right)}{\sqrt{2\pi\sigma^2}} \]

\[ M_n = \text{Mean} \]

\[ \sigma^2 = N_o B \quad (\text{Variance}) \]

So, IF \( G(f) \) IS

"White" or "Bandlimited White"

AND a SAMPLE of the process

follows the Gaussian Probability Law, we say the process

IS a WHITE OR A BANDLIMITED WHITE GAUSSIAN RANDOM PROCESS.
\[ x(\mu, t) \]
\[ \mu_0, \mu \]
\[ \sigma^2(t) \]

So if \( G(f) \) is

"white" or "band-limited white"

\( X(\mu_n, t) \)

\[ x(\mu, t) \]

\[ \text{ARV} \]

WHITE Gaussian Random Process

\[ \text{Follows the Gaussian Probability Law} \]
WHERE

\[ K = \text{Boltzmann's Constant} \]
\[ T = \text{Absolute Temp} \]

@ \( T_0 = 290^\circ K \)

\[ H_0 = -174 \text{ J mol}^{-1} \text{ Hz}^{-1} \]

REF: Sklar \"Random Signals\"