(we will use the abbreviation QL to refer to problems on this list
the list includes questions from prior midterm and final exams)

VECTORS AND MATRICES

1. Pages 27-30 of the text, Problem 1.4.8
2. Pages 27-30 of the text, Problem 1.4.11
3. Pages 27-30 of the text, Problem 1.4.12
4. Pages 27-30 of the text, Problem 1.4.20
5. Pages 27-30 of the text, Problem 1.4.21
6. Pages 27-30 of the text, Problem 1.4.22
7. Pages 27-30 of the text, Problem 1.4.24
8. The matrix $A$ as well as it’s inverse $A^{-1}$ are given below:

\[
A = \begin{bmatrix}
1 & -1 & 0 \\
1 & 0 & -1 \\
-6 & 2 & 3
\end{bmatrix}
\quad \text{and} \quad
A^{-1} = \begin{bmatrix}
-2 & -3 & -1 \\
-3 & -3 & -1 \\
-2 & -4 & -1
\end{bmatrix}.
\]

The matrix

\[
B = \begin{bmatrix}
1 & 0 & -1 \\
1 & -1 & 0 \\
-6 & 2 & 3
\end{bmatrix}
\]

is obtained by interchanging the first two rows of $A$. Write down the inverse of $B$.  
Hint: Write $B = PA$ for a suitable matrix $P$.


10. Exercise 1.5.6 of the text, page 40.
GAUSIAN ELIMINATION AND LDU FACTORIZATION

11. Express the matrix

\[
A = \begin{bmatrix}
0 & 0 & 4 \\
-6 & 6 & -12 \\
3 & 0 & 9
\end{bmatrix}
\]

in the form \( PA = LDW \) where

- \( P \) is a \((3 \times 3)\) permutation matrix
- \( L \) is a \((3 \times 3)\) lower triangular matrix with 1’s along the diagonal
- \( D \) is a diagonal matrix
- \( W \) is a \((3 \times 3)\) upper triangular matrix with 1’s along the diagonal ,

i.e., find the matrices \( P, L, D \) and \( W \).

12. The equation below describes the row-reduction of an \( n \times n \) matrix \( A \) using elementary row operations:

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & -2 & 1
\end{bmatrix}_{E_{32}}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-2 & 0 & 1
\end{bmatrix}_{E_{31}}
\begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}_{E_{21}}
\begin{bmatrix}
1 & 1 & 1 \\
-1 & 0 & 2 \\
2 & 4 & 6
\end{bmatrix}_{A}
= \begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & 3 \\
0 & 0 & -2
\end{bmatrix}_{U}
\]

Write down a \((3 \times 3)\) matrix \( L \) such that \( A = LU \).

13. Exercise 1.5.11, page 40 of the text.

**Cholesky decomposition, triangular matrices, singularity, inverse Computation**

14. Is the matrix

\[
A = \begin{bmatrix}
1 & -2 & 1 & 0 \\
6 & -3 & 16 & -4 \\
2 & 4 & 7 & -2 \\
3 & -6 & 6 & 2
\end{bmatrix}
\]

singular ?

15. Use the Gauss-Jordan method to find the inverse of the matrix

\[
A = \begin{bmatrix}
0 & -1 & 1 \\
2 & 1 & 4 \\
5 & 3 & 9
\end{bmatrix}
\]
16. “A typical Chinese problem, taken from the Han dynasty text *Nine Chapters of the Mathematical Art* (about 200 B.C.) reads: There are three classes of corn, of which three bundles of the first class, two of the second, and one of the third, make 39 measures. Two of the first, three of the second and one of the third, make 34 measures. And one of the first, two of their second and three of the third make 26 measures. How many measures of grain are contained in one bundle of each class? The system of equations here is:

\[
\begin{align*}
3x + 2y + z &= 39 \\
2x + 3y + z &= 34 \\
x + 2y + 3z &= 26
\end{align*}
\]

- Taken from notes of Victor Katz appearing in *Linear Algebra* by J. B. Fraleigh and R. A. Beauregard. Solve the problem using the augmented matrix formulation.

17. Factor the symmetric matrix

\[
A = \begin{bmatrix}
13 & 10 \\
10 & 8
\end{bmatrix}
\]

into a product \( A = SS^T \) (Cholesky decomposition), where \( S \) is a lower triangular matrix.

18. Exercise 1.7.6, page 59 of the text.

**Vector Spaces, subspaces**

19. Let \( V \) be a vector space. Show that if \( \mathbf{u} \) is in \( V \) and \( r \) is a scalar and if \( r\mathbf{u} = \mathbf{0} \), then either \( r = 0 \) or else, \( \mathbf{u} = \mathbf{0} \).

20. Identify geometrically, as clearly as you can, the subset of 3-dimensional (Euclidean) space \( \mathbb{R}^3 \) that corresponds to the column space of the matrix

\[
A = \begin{bmatrix}
0 & 2 \\
3 & 5 \\
4 & 0
\end{bmatrix}
\]

21. Let \( S \) be the subset of the \((x,y)\)-plane (i.e., the subset of \( \mathbb{R}^2 \)) given by

\[S = \{(x, y) \mid x + 4y = 13\}.

Is \( S \) a subspace of the \((x,y)\)-plane?
22. Consider the vector space $V = \mathbb{R}^2$ with addition $\oplus$ defined by

$[x, y] \oplus [w, z] = [x + w + 1, y + z - 2]$ and scalar multiplication $\odot$ defined by

$a \odot [x, y] = [ax + a - 1, ay - 2a + 2]$.

Does $V$ with these operations, form a vector space? If your answer is yes, identify the identity element and the additive inverse of each element. If your answer is no, explain clearly, why you feel this fails to be a vector space. Start by stating all the axioms that define a vector space and your conclusions on each of the axioms. (CARE - Do not jump to conclusions on this problem).

23. Which of the following are examples of subspaces? (In each case, the operations of vector addition and scalar multiplication are inherited from the parent vector space).

(a) the set

$\{ \underline{w} + w \mid w \in W \}$,

where $\underline{w}$ is a fixed nonzero element of $\mathbb{R}^n$ and $W$ is a fixed subspace of $\mathbb{R}^n$,

(b) the set of all $(3 \times 3)$ nonsingular matrices

(c) the set of all polynomials of the form

$c_0 + c_1X + c_2X^2 + X^3$

where $c_0, c_1, c_2$ are real numbers,

(d) the set of all odd functions $f : \mathbb{R} \to \mathbb{R}$, i.e., functions $f(x)$ such that

$f(-x) = -f(x) \; \forall x \in \mathbb{R}$.

(e) none of the above

24. Let $H$ be the matrix

$H = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 \\
\end{bmatrix}$

Assume that you are working in the field $F$ of two elements $\{0, 1\}$. Addition, denoted by $\oplus$, operates as follows

$0 \oplus 0 = 0 \quad 0 \oplus 1 = 1$

$1 \oplus 0 = 1 \quad 1 \oplus 1 = 0$
Multiplication of elements in the field, denoted by \( \otimes \), operates in the usual way, i.e.,

\[
\begin{align*}
0 \otimes 0 &= 0 & 0 \otimes 1 &= 0 \\
1 \otimes 0 &= 0 & 1 \otimes 1 &= 1
\end{align*}
\]

Determine all the vectors in the nullspace of the matrix \( H \). (Hint: the nullspace is in this case a finite set!)

**Columnspace, nullspace, rowspace and SLE**

25. Which of the following are examples of subspaces? (In each case, the operations of vector addition and scalar multiplication are inherited from the parent vector space).

   (a) the set
   \[
   \{ \mathbf{c} + \mathbf{w} \mid \mathbf{w} \in W \},
   \]
   where \( \mathbf{c} \) is a fixed nonzero element of \( \mathbb{R}^n \) and \( W \) is a fixed subspace of \( \mathbb{R}^n \),

   (b) the set of all \((3 \times 3)\) nonsingular matrices

   (c) the set of all polynomials of the form
   \[
   c_0 + c_1 X + c_2 X^2 + X^3
   \]
   where \( c_0, c_1, c_2 \) are real numbers,

   (d) the set of all odd functions \( f : \mathbb{R} \to \mathbb{R} \), i.e., functions \( f(x) \) such that
   \[
   f(-x) = -f(x) \quad \forall x \in \mathbb{R}.
   \]

   (e) none of the above

26. Identify geometrically, as clearly as you can, the subspaces of 3-dimensional (Euclidean) space \( \mathbb{R}^3 \) that correspond (i) to the rowspace and (ii) to the nullspace of the matrix:

   \[
   A = \begin{bmatrix}
   1 & 2 & -3 \\
   2 & -1 & 4 \\
   4 & 3 & -2
   \end{bmatrix}.
   \]

   Do you notice any relationship between the two?

27. Exercise 2.4.5, page 100 of the text. (You may ignore the part of the question that is within parenthesis).

28. For the matrix given in Review Exercise 2.4, page 128 of the text, determine the echelon form. (You are not required to find the dimension of its four fundamental subspaces).
29. How many possible patterns can you find (like the one in Fig. 2.2, page 72 of the text) for \((2 \times 3)\) echelon matrices. Entries to the right of the pivots are irrelevant.

30. Find necessary and sufficient conditions on the components \(b_1, b_2, b_3\) and \(b_4\) under which the system of linear equations

\[
\begin{bmatrix}
1 & 2 & 1 & 1 \\
2 & 1 & 0 & 3 \\
1 & -1 & -1 & 2 \\
0 & 3 & 2 & -1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} =
\begin{bmatrix}
b_1 \\
b_2 \\
b_3 \\
b_4
\end{bmatrix}
\]

will have at least one solution.

**SLE – the general case, linear independence**

31. Find conditions on \(b_1, b_2, b_3\) such that the system of linear equations

\[
\begin{align*}
2x + 2y + z + 7w &= b_1 \\
x + y + 2z + 8w &= b_2 \\
x + y + 0z + 2w &= b_3
\end{align*}
\]

will have at least one solution. Given that \(b_1 = -6, b_2 = 0, \) choose \(b_3\) such that the system of linear equations has at least one solution (it turns out that this is possible in this case). Then for this choice of \(b_1, b_2, b_3\), find all solutions to the system of linear equations. Follow the steps outlined in class: (i) first identify the augmented matrix; then reduce the left-hand side of this augmented matrix to echelon form; find the condition(s) for existence of a solution. Then choose \(b_3\) appropriately, identify a particular solution and the nullspace of the coefficient matrix and finally, write down an expression for the set of all solutions.

32. Row-reduce to echelon form the coefficient matrix of the system of linear equations:

\[
\begin{align*}
x_1 - 2x_2 + x_3 + 2x_4 &= 3, \\
x_1 + x_2 - x_3 + x_4 &= 2, \\
3x_1 + 6x_2 - 5x_3 + 2x_4 &= 5.
\end{align*}
\]

and hence find the most general solution. (Follow the steps outlined in Problem 1 above).

33. Are the vectors

\[
\begin{align*}
\vec{v}_1 &= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \\
\vec{v}_2 &= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \\
\vec{v}_3 &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \text{ and } \\
\vec{v}_4 &= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}
\end{align*}
\]

in \(\mathbb{R}^4\) linearly independent? Show all intermediate steps and explain your answer.
34. If the vectors \( \mathbf{u}, \mathbf{v} \) and \( \mathbf{w} \) of a vector space \( V \) are linearly independent, does it necessarily follow that the vectors \( \mathbf{u} + \mathbf{v}, \mathbf{v} + \mathbf{w} \) and \( \mathbf{w} + \mathbf{u} \) will always be linearly independent?

35. For the system of linear equations presented in Exercise 2.2.8, page 78 of the text, after finding an equation that \( c \) must satisfy in order for there to be a solution, use this equation to identify geometrically, the column space of the coefficient matrix \( A \).

36. Exercise 2.2.13, page 79 of the text. (Hint: first determine the coefficient matrix based on the given nullspace and the condition \( b_1 + b_2 = b_3 \) for existence of a solution. Then find a right-hand side for which \( [1 \ 1 \ 0]^T \) is a particular solution.)

**Spanning, Basis, coordinate representation, change of basis**

37. Exercise 2.3.7, page 87 of the text.

38. Find a basis each for the rowspace, columnspace and nullspace of the matrix

\[
A = \begin{bmatrix}
1 & 2 & 0 & -1 \\
1 & 3 & 1 & 1 \\
2 & 5 & 1 & 0 \\
3 & 6 & 0 & 0 \\
\end{bmatrix}.
\]

39. Find a basis for the vector space consisting of all polynomials \( f(x) = a_0 + a_1x + a_2x^2 \) of degree less than or equal to 2 that satisfy in addition, the condition that their coefficients sum to zero, i.e., \( a_0 + a_1 + a_2 = 0 \).

40. What is the coordinate representation of the vector \( [1 \ 0 \ 0]^T \) with respect to the basis

\[
\mathcal{A} = \{[1 \ 2 \ 3]^T, [0 \ 1 \ 2]^T, [0 \ 0 \ 1]^T\}?
\]

41. Consider the following two basis for \( \mathbb{R}^2 \):

\[
\mathcal{A} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}
\]

and

\[
\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}
\]

Find a matrix \( P \) such that

\[
[x]_\mathcal{B} = P[x]_\mathcal{A}
\]

for all \( x \in \mathbb{R}^2 \).
42. Consider the following two basis for the vector space of all $(2 \times 2)$ upper triangular matrices $U$:

\[ A = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\} \]

and

\[ B = \left\{ \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \]

Find a matrix $P$ such that

\[ [U]_B = P[A]_A \]

for all upper triangular matrices $U$.

**Dimension, Rank**

43. The nullspace of a certain $(3 \times 3)$ matrix is a plane in $\mathbb{R}^3$ that passes through the origin. What is the rank of $A$?

44. The $(4 \times 3)$ matrix $A$ has rank 3. Which of the following is true of $A$?

(a) If a solution to $Ax = b$ exists, then that solution is unique.
(b) $Ax = b$ always has at least one solution
(c) Neither of the above two statements is necessarily true

45. Let $A$ be an $(m \times n)$ matrix and consider the system of linear equations $Ax = b$. Let $B = [A \mid \begin{bmatrix} b \end{bmatrix}]$ be the $(m \times n + 1)$ augmented matrix. Then $Ax = b$ has at least one solution if and only if the matrices $A$ and $B$ have the same rank.

(i) True  
(ii) False

46. What is the dimension of the vector space of all $3 \times 3$ upper triangular matrices? (A is upper triangular if and only if $a_{21} = a_{31} = a_{32} = 0$).

47. Exercise 2.18, page 129 of the text.

48. Exercise 2.21, page 129 of the text.

49. Reduce the matrix $A$ below to echelon form. Use the echelon form to indicate with a ($\checkmark$) all the statements below that are true concerning the matrix $A$.

\[ A = \begin{bmatrix} 1 & -2 & 2 & -1 \\ -3 & 6 & 1 & 10 \\ 1 & -2 & -4 & -7 \end{bmatrix} \]

(a) $A$ has a left inverse
(b) $A$ has a right inverse
(c) For every $\underline{b} \in \mathbb{R}^3$, $A\underline{x} = \underline{b}$ has a solution.
(d) If $A\underline{x} = \underline{b}$ has a solution, then that solution is unique.
(e) none of the above is true of $A$.


**Geometric Notions, Least-squares Approximation**

51. Let $A$ be an $(m \times n)$ matrix and $\underline{b}$ be an $(m \times 1)$ vector.

   (a) Under what conditions on $A$ does the equation $A^t A\underline{x} = A^t \underline{b}$ have a solution $\underline{x}$?
   (b) Under what conditions on $A$ does the equation $A^t A\underline{x} = A^t \underline{b}$ have a unique solution $\underline{x}$ (given that a solution exists)?

52. Let $P$ denote the plane in $\mathbb{R}^3$ containing the vectors $\underline{z} = [1 1 1]^t$ and $\underline{y} = [1 2 3]^t$. Find a vector $\underline{z}$ in $P$ that is orthogonal to $\underline{y}$.

53. The result of projecting the vector $\underline{z} = [2 6 1]^t$ onto the plane $Q$ in $\mathbb{R}^3$ is the vector $\underline{z} = [1 5 3]^t$. Identify as clearly and as explicitly as you can, the plane $Q$.

54. Find the quadratic polynomial $y = ax^2 + bx + c$ that is the best least-squares fit to the points
   
   $(-2, -1), \; (-1, 2), \; (0, 4), \; (1, 1), \; (2, -3)$.

55. Exercise 3.2.8, page 151 of the text.

56. Exercise 3.2.7, page 151 of the text.

57. Exercise 3.2.3, page 151 of the text.

**Spanning, Basis, coordinate representation, change of basis**

58. Exercise 2.3.7, page 87 of the text.

59. Find a basis each for the rowspace, columnspace and nullspace of the matrix

   \[
   A = \begin{bmatrix}
   1 & 2 & 0 & -1 \\
   1 & 3 & 1 & 1 \\
   2 & 5 & 1 & 0 \\
   3 & 6 & 0 & 0 \\
   \end{bmatrix}
   \]
60. Find a basis for the vector space consisting of all polynomials \( f(x) = a_0 + a_1 x + a_2 x^2 \) of degree less than or equal to 2 that satisfy in addition, the condition that their coefficients sum to zero, i.e., \( a_0 + a_1 + a_2 = 0 \).

61. What is the coordinate representation of the vector \( [1 \ 0 \ 0]^T \) with respect to the basis \( \mathcal{A} = \{ [1 \ 2 \ 3]^T, [0 \ 1 \ 2]^T, [0 \ 0 \ 1]^T \} \)?

62. Consider the following two basis for \( \mathbb{R}^2 \):
\[
\mathcal{A} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}
\]
and
\[
\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}
\]
Find a matrix \( P \) such that
\[
[x]_B = P[x]_A
\]
for all \( x \in \mathbb{R}^2 \).

63. Consider the following two basis for the vector space of all \((2 \times 2)\) upper triangular matrices \( U \):
\[
\mathcal{A} = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}
\]
and
\[
\mathcal{B} = \left\{ \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}
\]
Find a matrix \( P \) such that
\[
[U]_B = P[U]_A
\]
for all upper triangular matrices \( U \).

**LSA, orthogonality, Gram-Schmidt process, complex vectors**

64. Find the least-squares approximate solution of the overdetermined system:
\[
\begin{bmatrix}
3 & 1 \\
1 & 2 \\
2 & -1
\end{bmatrix}
\begin{bmatrix}
x
\end{bmatrix}
= \begin{bmatrix}
1 \\
0 \\
-2
\end{bmatrix}.
\]
65. Apply the Gram-Schmidt orthogonalization process and derive an orthonormal basis for \( \mathbb{R}^3 \), starting with the three vectors

\[
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}, \quad
\begin{bmatrix}
0 \\
1 \\
1
\end{bmatrix}, \quad
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}.
\]

66. Consider the least-squares approximation problem \( Ax \approx b \) where

\[
A = \begin{bmatrix}
1 & -6 \\
3 & 6 \\
4 & 8 \\
5 & 0 \\
7 & 8
\end{bmatrix}
\]

and \( b = [-3, 7, 1, 0, 4]^T \). Use the Gram-Schmidt orthogonalization process to replace the second column vector of the matrix \( A \) by a vector that is orthogonal to the first column vector. Let \( B \) denote the new matrix. Solve \( Ax \approx b \) by solving instead \( Bx \approx b \) instead. Why is this an easier problem? Why is this procedure justified?

67. Exercise 3.1.9 of the text, page 142.

68. Let \( w_i, \ i = 1, 2, 3, 4 \), denote the four column vectors of the matrix

\[
A = \frac{1}{2} \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1
\end{bmatrix}.
\]

Note that the four column vectors are orthogonal and therefore, linearly independent.

(a) Express the vector \( x = [0, 3, 6, 12]^T \) as a linear combination of the four column vectors.

(b) Compute the inner products \( x^T w_i, \ i = 1, 2, 3, 4 \).

(c) Are the above two results related? Can you generalize these results?

**COMPLEX VECTORS. EIGENVALUES AND EIGENVECTORS**

69. Find the lengths of the vectors \( a = [1, 2, i]^T \), \( b = [1 + i, 1 - 2i, 3 + 4i]^T \) as well as their inner product. Find a vector \( [x, y, z]^T \) that is orthogonal to both vectors.

70. Find the eigenvalues of a \( (2 \times 2) \) matrix \( A \) whose trace equals 2 and whose determinant equals \(-3\).
71. The first two terms of a sequence are \( a_0 = 0 \) and \( a_1 = 1 \). Subsequent terms are generated using
\[
a_k = 2a_{k-1} + a_{k-2}, \text{ for } k \geq 2.
\]
Find \( a_\infty \).

72. Determine whether or not the matrix
\[
A = \begin{bmatrix}
1 & -3 & 3 \\
0 & -5 & 6 \\
0 & -3 & 4
\end{bmatrix}.
\]
is diagonalizable. If \( A \) is diagonalizable, express \( A \) in the form \( A = E \Lambda E^{-1} \), where \( \Lambda \) is a diagonal matrix.
PRIOR MIDTERM EXAMS

EE441
KUMAR

• unless otherwise specified, all vectors are column vectors
• on some multiple-choice questions, more than one choice may be correct; unless otherwise instructed, indicate with a (√) all the correct answers you can spot
• multiple-choice questions will be graded based only upon your answer
• clearly identify all your final answers

Sp 1998

73. Let $V$ be the vector space of all functions $f$ from $\mathbb{R}$ to $\mathbb{R}$. Which (if any) of the following sets of functions are subspaces of $V$?

- the subset of all $f$ such that $f(x^2) = (f(x))^2$
- the subset of all $f$ such that $f(0) = f(1)$
- the subset of all $f$ such that $f(-1) = 0$

2 points

74. Consider two subspaces $V$ and $W$ of $\mathbb{R}^3$.

(a) Is the intersection $V \cap W$ necessarily a subspace of $\mathbb{R}^3$?
(b) Is the union $V \cup W$ necessarily a subspace of $\mathbb{R}^3$?

2 points

75. Consider a linear system of four equations with three unknowns. We are told that the system has a unique solution. What does the reduced-row echelon form of the coefficient matrix of this system look like? Use \( \bigcirc \) to denote a pivot and \( \ast \) to denote an entry that may or may not be zero.

2 points

76.

(a) Let $A$ be a $(4 \times 4)$ matrix and let $\overrightarrow{b}$ and $\overrightarrow{c}$ be two vectors in $\mathbb{R}^4$. We are told that the system $A\overrightarrow{x} = \overrightarrow{b}$ is inconsistent, i.e., does not have a solution. What can you say about the number of solutions of the system $A\overrightarrow{x} = \overrightarrow{c}$? Be as precise as you can possibly be in your answer.
(b) Let $A$ be a $(4 \times 3)$ matrix and let $b$ and $c$ be two vectors in $\mathbb{R}^4$. We are told that the system $Ax = b$ has a unique solution. What can you say about the number of solutions of the system $Ax = c$? Be as precise as you can possibly be in your answer.

2 points

77. If the $n$ columns of an $n \times n$ matrix $A$ are linearly dependent, the same will always be true of the $n$ columns of the matrix $A^2$.

(i) True (ii) False

2 points

78. Singularity of the $n \times n$ matrix $A$ implies that

- the matrix $A$ does not have an inverse
- for every $b$ in $\mathbb{R}^n$, the system of equations $Ax = b$ always has an infinite number of solutions.
- the columns of $A$ are linearly dependent
- none of the above

Identify all the correct implications you can spot.

2 points

79. The equation below describes the row-reduction of a $3 \times 3$ matrix $A$ using elementary row operations:

$$
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0
\end{bmatrix}
\cdot
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-1 & 0 & 1
\end{bmatrix}
\cdot
\begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\cdot
\begin{bmatrix}
0 & 0 & 1 \\
1 & 2 & 1 \\
0 & -1 & 0
\end{bmatrix}
= 
\begin{bmatrix}
1 & 2 & 1 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{bmatrix}
.$$ 

Determine the matrices $P$, $D$ and $W$ in the $PA = LDW$ decomposition of $A$.

4 points

80. Consider the linear system

$$
\begin{align*}
x + y - z &= 2 \\
x + 2y + z &= 3 \\
x + y + (k^2 - 5)z &= k
\end{align*}
$$

14
where \( k \) is an arbitrary constant. For which choice(s) of \( k \) does this system have a unique solution? For which choice(s) of \( k \) does the system have infinitely many solutions? For which choice(s) of \( k \) is the system inconsistent?

4 points

81. Identify all cubic polynomials (i.e., polynomials of degree 3) of the form \( f(x) = a_3x^3 + a_2x^2 + a_1x + a_0 \) for some real coefficients \( a_i \) such that the graph \( y = f(x) \) passes thru the two points \((1, 0)\) and \((2, -15)\) in the \((x, y)\)-plane.

6 points

82. Let \( V \) be the vector space of all functions \( f : \mathbb{R} \to \mathbb{R} \). Let \( f_1, f_2 \) and \( f_3 \) be functions satisfying

\[
\begin{align*}
  f_1(x_1) &= 2, \quad f_1(x_2) = 0, \quad f_1(x_3) = 0 \\
  f_2(x_1) &= 3, \quad f_2(x_2) = 4, \quad f_2(x_3) = 0 \\
  f_3(x_1) &= 2, \quad f_3(x_2) = 7, \quad f_3(x_3) = 9
\end{align*}
\]

Are the functions \( f_1, f_2, f_3 \) linearly independent (i.e., is there a nontrivial linear combination of these functions that will yield the function representing the zero vector in \( V \))? Explain your answer and show all intermediate steps clearly.

4 points

83. Let \( V \) and \( W \) be subspaces of \( \mathbb{R}^3 \) spanned by 3 and 2 vectors respectively, as shown below:

\[
V = \langle \begin{bmatrix} 1 \\ 0 \\ 2 \\ \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 12 \\ \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 8 \\ \end{bmatrix} \rangle, \quad \text{and} \quad W = \langle \begin{bmatrix} 1 \\ 1 \\ \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ \end{bmatrix} \rangle.
\]

Is \( W \) a subspace of \( V \)? Explain your answer, showing all intermediate steps.

6 points

Fall 1995

84. If \( A, B, C \) and \( D \) are all \((3 \times 2)\) matrices and \( E, F \) are \((2 \times 2)\) matrices, is it then always true that

\[
\begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} E \\ F \end{bmatrix} = \begin{bmatrix} AE + BF \\ CE + DF \end{bmatrix}?
\]

2 points

(i) True \quad (ii) False
85. Let $A$ be a $(n \times n)$ square matrix with precisely one nonzero element in each row and precisely one nonzero element in each column of $A$. Does this guarantee that $A$ has an inverse?

2 points

(i) True  (ii) False

86. The equation below describes the row-reduction of a $3 \times 3$ matrix $A$ using elementary row operations:

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & -2 & 1
\end{bmatrix}
\cdot
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-1 & 0 & 1
\end{bmatrix}
\cdot
\begin{bmatrix}
1 & 0 & 0 \\
-3 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\cdot
\begin{bmatrix}
1 & -3 & 1 \\
3 & -8 & 2 \\
1 & -1 & -2
\end{bmatrix}
= \begin{bmatrix}
1 & -3 & 1 \\
0 & 1 & -1 \\
0 & 0 & -1
\end{bmatrix}.
\]

Write down a $(3 \times 3)$ matrix $L$ such that $A = LU$.

2 points

Ans $L =$

87. Does the subset

\[W = \{(x, y) | x, y, \in \mathbb{R}, \text{ and } x, y \geq 0 \text{ or } x, y \leq 0\}\]

i.e., the union of the first and third quadrants of the $(x, y)$-plane $\mathbb{R}^2$, constitute a subspace (over the real numbers) of the $(x, y)$-plane?

(i) True  (ii) False

2 points

88. Is the matrix

\[B = \begin{bmatrix}
0 & 1 & 14 & 3 \\
0 & 0 & -2 & 2 \\
0 & 0 & 0 & 14 \\
0 & 0 & 0 & 0
\end{bmatrix}\]
in row-reduced echelon-form? (In other words, does there exist a matrix $A$ which upon row-reduction to echelon form would yield $B$?)

(i) Yes  
(ii) No  

2 points

89. Let $A$ be the matrix

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 3 & 5 \\ 1 & 1 & 11 \\ 1 & 1 & 5 \end{bmatrix}.$$ 

Which of the following is true of $A$?

(a) If a solution to $A \mathbf{x} = \mathbf{b}$ exists, then that solution is unique
(b) $A \mathbf{x} = \mathbf{b}$ always has at least one solution for every $(4 \times 1)$ vector $\mathbf{b}$
(c) Neither of the above two statements is necessarily true

2 points

90. The four vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ lie in $\mathbb{R}^4$. It is known that no vector in the set of 4 vectors can be expressed as a linear combination of the remaining three vectors. Does it then follow that the set

$$\{ \mathbf{a} , \mathbf{b} , \mathbf{c} , \mathbf{d} \}$$

forms a set of four linearly independent vectors?

(i) True  
(ii) False  

2 points

91. Does the vector

$$\mathbf{x} = \begin{bmatrix} -1 \\ -1 \\ -3 \end{bmatrix}$$

belong to the subspace of 3-dimensional space $\mathbb{R}^3$ spanned by the vectors

$$\mathbf{a} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}?$$
(i) Yes  
(ii) No  

2 points

92. Identify \textit{precisely} all conditions on $a, b, c, d$ under which the columns of

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

are linearly independent. Before you begin, explain your approach in a brief sentence.

4 points

93. Use the Gauss-Jordan method to find the inverse of the matrix

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ -6 & 2 & 3 \end{bmatrix}.$$

4 points

94. Consider the system of linear equations

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 1 & 3 & 1 & 1 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}. $$

The augmented matrix was row reduced to echelon form as shown below:

$$\begin{bmatrix} 1 & 2 & 0 & -1 & a \\ 1 & 3 & 1 & 1 & b \\ 2 & 5 & 1 & 0 & c \\ 3 & 6 & 0 & 0 & d \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 0 & -1 & a \\ 0 & 1 & 2 & b-a \\ 0 & 0 & 3 & d-3a \\ 0 & 0 & 0 & c-b-a \end{bmatrix}. $$

(a) Identify all values of $a, b, c, d$ under which this system of linear equations will have at least one solution.

2 points

(b) Identify the free variables.

1 point
(c) Find a particular solution for the case, \( a = 2, b = -2, c = 0, d = 6 \).

(d) Find a basis for the nullspace of \( A \). (\( A \) is the \( 4 \times 4 \) coefficient matrix as usual).

(e) Find a basis for the rowspace of \( A \).

Spring 1994

95. Identify with a \( \sqrt{\ } \) all the answers that are correct:

- If the first and third rows of \( A \) are the same, so are the first and third rows of \( AB \)
- If the first and third rows of \( B \) are the same, so are the first and third rows of \( AB \)
- If the square matrix \( C \) has an inverse, then \( (C^3)^{-1} = (C^{-1})^3 \)

96. The \( (2n \times 2n) \) matrix \( M \) is given by

\[
M = \begin{bmatrix}
A & 0 \\
0 & B
\end{bmatrix}
\]

where \( A \) and \( B \) are a pair of invertible \( (n \times n) \) matrices and \( 0 \) denotes the \( (n \times n) \) all-zero matrix. Is it then true that

\[
M^{-1} = \begin{bmatrix}
A^{-1} & 0 \\
0 & B^{-1}
\end{bmatrix}
\]

(i) Yes  
(ii) No

97. The equation below describes the row-reduction of an \( n \times n \) matrix \( A \) using elementary row operations:

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & -2 & 1
\end{bmatrix}_{\mathcal{E}_{22}} \cdot
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-2 & 0 & 1
\end{bmatrix}_{\mathcal{E}_{31}} \cdot
\begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}_{\mathcal{E}_{21}} \cdot
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 0 \\
-1 & 0 & 2
\end{bmatrix}_A = \begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & 3 \\
0 & 0 & -2
\end{bmatrix}_U.
\]

Write down a \( (3 \times 3) \) matrix \( L \) such that \( A = LU \).
Ans I. =

98. Identify geometrically, as clearly as you can, the subset of 3-dimensional (Euclidean) space \( \mathbb{R}^3 \) that corresponds to the column space of the matrix

\[
A = \begin{bmatrix}
2 & 0 \\
0 & 1 \\
2 & 0
\end{bmatrix}.
\]

99. The matrix \( A \) is an \((m \times n)\) matrix with \( n > m \). Then

- the equation \( Ax = 0 \) always has at least one non-zero solution
- the rows of \( A \) are always dependent
- the columns of \( A \) are always dependent
- none of the above

Indicate with a \( \checkmark \) the statement(s) you think is (are) correct.

100. The vector space \( V \) has a basis containing exactly 4 vectors. Then

- any 3 vectors in \( V \) are linearly independent,
- any 5 vectors span \( V \),
- any 5 vectors in \( V \) are linearly dependent,
- None of the above is necessarily true.

101. Let \( S \) be the subset of the \((x,y)\)-plane (i.e., the subset of \( \mathbb{R}^2 \)) given by

\[
S = \{(x, y) \mid y = 3x + 1\}.
\]

Is \( S \) a subspace of the \((x,y)\)-plane?
102. Reduce the matrix \( A \) below to echelon form. Use the echelon form to indicate with a \( \checkmark \) all the statements below that are true concerning the matrix \( A \).

\[
A = \begin{bmatrix}
1 & -2 & 2 & -1 \\
-3 & 6 & 1 & 10 \\
1 & -2 & -4 & -7
\end{bmatrix}.
\]

(a) \( A \) has a left inverse
(b) \( A \) has a right inverse
(c) For every \( \mathbf{b} \in \mathbb{R}^3 \), \( A\mathbf{x} = \mathbf{b} \) has a solution.
(d) If \( A\mathbf{x} = \mathbf{b} \) has a solution, then that solution is unique.
(e) none of the above is true of \( A \).

103. Elementary row-reduction using the Gaussian elimination procedure, (no row interchanges were needed) of the \((2 \times 2)\) symmetric matrix \( A \) produced the upper triangular matrix \( U \) shown below:

\[
U = \begin{bmatrix}
1 & 3 \\
0 & 1
\end{bmatrix}.
\]

(Thus, \( A = LU \) for some triangular matrix \( L \) with 1’s along the diagonal). Find the matrix \( A \).

\[\text{Ans } A = \]

104. Use the Gauss-Jordan method to find the inverse of the matrix

\[
A = \begin{bmatrix}
1 & 2 & 0 \\
2 & 4 & 1 \\
1 & 3 & 2
\end{bmatrix}.
\]
Show all intermediate steps in your working clearly.

6 points

Ans. $A^{-1} =$

105. Extend the pair of vectors

$$
\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix}
$$

to a basis for $\mathbb{R}^3$, i.e., find a vector $\mathbf{c}$ such that $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ are a set of linearly independent vectors.

4 points

Before proceeding to solve the problem, explain in one or two sentences, your approach to the problem.

2 points

106. Are the vectors

$$
\mathbf{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{w}_4 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}
$$

in $\mathbb{R}^4$ linearly independent? Show all intermediate steps and explain your answer. 6 points

Sp 1993

107. Let $A$ be the $(m \times n)$ matrix:

$$
A = \begin{bmatrix}
\mathbf{u}_1^T \\
\mathbf{u}_2^T \\
\vdots \\
\mathbf{u}_m^T
\end{bmatrix} = \begin{bmatrix}
\mathbf{w}_1 & \mathbf{w}_2 & \cdots & \mathbf{w}_n
\end{bmatrix}.
$$

Let $B$ be the $(n \times p)$ matrix

$$
B = \begin{bmatrix}
\mathbf{y}_1^T \\
\mathbf{y}_2^T \\
\vdots \\
\mathbf{y}_p^T
\end{bmatrix} = \begin{bmatrix}
\mathbf{z}_1 & \mathbf{z}_2 & \cdots & \mathbf{z}_p
\end{bmatrix}.
$$
Write down a mathematical expression for the matrix product $AB$ in terms of (some or all of) the vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{y}$, $\mathbf{z}$ and their transposes.

2 points

108. A rectangular $(m \times n)$ matrix with $m \neq n$ that has a left inverse also has a right inverse and vice-versa.

(i) True 
(ii) False

2 points

109. The matrix

$$A = \begin{bmatrix} 1 & 4 & -2 & 7 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

- is singular
- is nonsingular
- cannot say – could be either singular or nonsingular.

2 points

110. The collection of all $(2 \times 2)$ matrices forms a vector space. Does the set of all nonsingular $(2 \times 2)$ matrices form a subspace of this vector space?

(i) True 
(ii) False

2 points

111. In a certain system $A \mathbf{x} = \mathbf{0}$ of homogeneous linear equations involving the four unknowns $x_1, x_2, x_3$ and $x_4$, $A$ is a $(3 \times 4)$ matrix and the basic variables $x_1$ and $x_3$ are related as follows to the free variables $x_2$ and $x_4$:

$$x_1 = x_2 - x_4 \text{ and } x_3 = 5x_2 + 2x_4.$$ 

Write down a basis for the nullspace of the matrix $A$.

2 points
112. Let $V$ be the vector space of all continuous functions defined over the real line (the set of all real numbers).

- Identify the zero vector in this vector space.
- Are the continuous functions $\cos(x)$ and $\sin(x)$, (where $x$ is in radians and varies over all the real numbers) linearly independent?

(i) True  (ii) False

2 points

113. Let $A$ be the matrix 

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 3 & 5 \\ 1 & 1 & 11 \\ 1 & 1 & 5 \end{bmatrix}.$$ 

Determine the rank of $A$.

1 point

Which of the following is true of $A$?

- If a solution to $A \bar{x} = \bar{b}$ exists, then that solution is unique
- $A \bar{x} = \bar{b}$ always has at least one solution
- Neither of the above two statements is necessarily true

1 point

114. Express the matrix 

$$A = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & 3 \\ -1 & 1 & -2 \end{bmatrix}$$

in the form $PA = LDU$ where 

- $P$ is a $(3 \times 3)$ permutation matrix
- $L$ is a $(3 \times 3)$ lower triangular matrix with 1’s along the diagonal
- $D$ is a diagonal matrix
- $U$ is a $(3 \times 3)$ upper triangular matrix with 1’s along the diagonal

i.e., find the matrices $P$, $L$, $D$ and $U$. 

24
115. Find necessary and sufficient conditions on the components $b_1, b_2, b_3$ and $b_4$ under which the system of linear equations

$$
\begin{bmatrix}
1 & 2 & 1 & 1 \\
2 & 1 & 0 & 3 \\
1 & -1 & -1 & 2 \\
0 & 3 & 2 & -1
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} = \begin{bmatrix}
b_1 \\
b_2 \\
b_3 \\
b_4
\end{bmatrix}
$$

will have at least one solution.

4 points

116. Consider the system of linear equations $A \mathbf{x} = \mathbf{b}$ where

$$
A = \begin{bmatrix}
1 & 1 & 0 & 0 \\
1 & 3 & 3 & 4 \\
1 & 1 & c & 1 \\
1 & 1 & 1 & d
\end{bmatrix}.
$$

Determine the set of all values of $c$ and $d$ for which this system of equations will have a unique solution.

6 points

117. Find a basis for the vector space

$$
W = \langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4 \rangle
$$

spanned by the $(4 \times 1)$ vectors

$$
\mathbf{a}_1 = \begin{bmatrix}
1 \\
3 \\
-1 \\
-2
\end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix}
2 \\
6 \\
-1 \\
-3
\end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix}
3 \\
8 \\
-3 \\
-5
\end{bmatrix} \text{ and } \mathbf{a}_4 = \begin{bmatrix}
-1 \\
-2 \\
1 \\
1
\end{bmatrix}.
$$

4 points

Fall 1992

118. An $(n \times n)$ lower triangular matrix has an inverse

(a) iff (if and only if) all the $n$ diagonal elements equal 1
(b) iff at least one diagonal element equals 1
(c) iff all the $n$ diagonal elements are nonzero
(d) always
(e) none of the above

2 points

119. Identify below with a (√) all the sets described below, that are vector spaces under the usual notion of addition and scalar multiplication:

(a) the set

$$\{ \underline{c} + \underline{w} \mid \underline{w} \in W \},$$

where $\underline{c}$ is a fixed nonzero element of $\mathbb{R}^n$ and $W$ is a fixed subspace of $\mathbb{R}^n$,

(b) the set of all $(3 \times 3)$ nonsingular matrices

(c) the set of all polynomials of the form

$$c_0 + c_1 X + c_2 X^2 + X^3$$

where $c_0, c_1, c_2$ are real numbers,

(d) the set of all odd functions $f : \mathbb{R} \to \mathbb{R}$, i.e., functions $f(x)$ such that

$$f(-x) = -f(x) \ \forall x \in \mathbb{R}.$$ 

(e) none of the above

2 points

120. Indicate with a (√) all the statements below that are true concerning the matrix

$$A = \begin{bmatrix} 2 & 4 & 3 & 6 \\ 0 & -1 & 0 & 12 \\ 0 & 0 & 0 & 5 \end{bmatrix}.$$ 

(a) $A$ has a left inverse

(b) $A$ has a right inverse

(c) For every $\underline{b} \in \mathbb{R}^3$, $A\underline{y} = \underline{b}$ has a solution.

(d) If $A\underline{y} = \underline{b}$ has a solution, then that solution is unique.

(e) none of the above is true of $A.$

26
121. Upon row-reduction, the \((m \times n)\) matrix \(A\) yields the row-reduced echelon matrix \(U\). Indicate below with a \((\checkmark)\) all the correct answers concerning \(A\), \(U\) and \(A^T A\) that you can spot:

(a) \(A\) and \(U\) have the same rowspace,
(b) \(A\) and \(U\) have the same column space,
(c) \(A\) and \(A^T A\) have the same rank,
(d) \(A\) and \(A^T A\) have the same column space,

122. Consider the linear transformation \(\mathcal{L}\) that projects each point \((x, y)\) in the \((x, y)\)-plane to the nearest point \(\mathcal{L}( (x, y) )\) on the straight line given by \(y = x\).

For example the point \(P\) in the figure above is projected onto the point \(Q\) lying on the straight line \(x = y\). Find a matrix \(A\) such that \(A \begin{bmatrix} x \\ y \end{bmatrix}\) is the vector whose coordinates are the coordinates of the point \(\mathcal{L}( (x, y) )\) for all points \((x, y)\) in the plane.

You may assume throughout, that the basis in use is the standard basis for \(\mathbb{R}^2\).

123. Identify a vector \(z\) in \(\mathbb{R}^4\) of the form \([z_1, z_2, 0, 0]^T\) which makes a 45° angle with the vector \([1, 1, 1, 1]^T\) lying in \(\mathbb{R}^4\) and which moreover, has unit length, i.e., \(z_1^2 + z_2^2 = 1\).
124. Given the matrices
\[
A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = A(A^T A)^{-1} A^T = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix},
\]
find the projection of the vector $[4 \ 5 \ 6]^T$ onto the column space of the matrix $B$.

2 points

125. The Gram-Schmidt procedure, applied to the columns $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ of the matrix $A$ given by
\[
A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix},
\]
resulted in the set $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4$ of orthonormal vectors which are the columns of the orthogonal matrix $Q$ given by
\[
Q = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3 & \mathbf{q}_4 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix}.
\]
In the associated $QR$ decomposition of the matrix $A$, determine the elements belonging to the third column of $R$, i.e., if
\[
A = Q \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ 0 & r_{22} & r_{23} & r_{24} \\ 0 & 0 & r_{33} & r_{34} \\ 0 & 0 & 0 & r_{44} \end{bmatrix},
\]
determine $r_{13}, r_{23}$ and $r_{33}$.

2 points

126. Express the matrix
\[
A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & 5 \\ 3 & 8 & 1 \end{bmatrix}
\]
in the form $PA = LDU$ where
- $P$ is a $(3 \times 3)$ permutation matrix
• \( L \) is a \((3 \times 3)\) lower triangular matrix with 1’s along the diagonal,
• \( U \) is a \((3 \times 3)\) upper triangular matrix with 1’s along the diagonal,
• \( D \) is a \((3 \times 3)\) diagonal matrix,

i.e., identify the matrices \( P, L, D \) and \( U \).

8 points

127. Given the \((4 \times 4)\) matrix

\[
A = \begin{bmatrix}
1 & 2 & 1 & -1 \\
2 & 6 & 5 & 2 \\
1 & 4 & 4 & 6 \\
3 & 8 & 6 & 1
\end{bmatrix},
\]

row-reduce \( A \) to an echelon matrix \( U \).

2 points

Next, use \( U \) to identify a basis for

(a) the column space of \( A \),

2 points

(b) the rowspace of \( A \)

2 points

(c) and the nullspace of \( A \).

3 points

128. Let \( \mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3, \mathbf{h}_4 \) be the linearly independent column vectors of the matrix

\[
H = \begin{bmatrix}
\mathbf{h}_1 & \mathbf{h}_2 & \mathbf{h}_3 & \mathbf{h}_4 \\
\end{bmatrix} = \begin{bmatrix}
7/25 & 0 & -24/25 & 0 \\
0 & 1 & 0 & 0 \\
-24/25 & 0 & -7/25 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]

(a) Compute \( H^T H \).

2 points

(b) Express \([1, 1, 1, 1]^T\) as a linear combination of the vectors \( \mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3, \mathbf{h}_4 \).

3 points
(c) Project $[1, 1, 1, 1]^T$ onto the space spanned by $b_3$ and $b_4$, i.e., onto the column space of the matrix

$$
A = \begin{bmatrix}
-24/25 & 0 \\
0 & 0 \\
-7/25 & 0 \\
0 & 1
\end{bmatrix}.
$$

3 points

---

**Fall 1988**

---

129. For every pair of square matrices $A$ and $B$ (of common size (nxn)), it is always true that

(a) $(AB)^k = A^kB^k$

(b) $(A^3)^{-1} = (A^{-1})^3$ whenever $A$ is invertible.

Indicate with a (✓) the statement(s) you think is(are) correct.

130. In the equation $AB = C$ shown below, $A$ and $C$ have been partitioned into blocks (submatrices). Show how you would similarly partition $B$ so as to express the blocks of $C$ in terms of the blocks of $A$ (and the appropriate blocks of $B$).

$$
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{14} \\
a_{21} & a_{22} & a_{23} & a_{24} \\
a_{31} & a_{32} & a_{33} & a_{34}
\end{bmatrix}
\begin{bmatrix}
b_{11} & b_{12} & b_{13} \\
b_{21} & b_{22} & b_{23} \\
b_{31} & b_{32} & b_{33} \\
b_{41} & b_{42} & b_{43}
\end{bmatrix}
= \begin{bmatrix}
c_{11} & c_{12} & c_{13} \\
c_{21} & c_{22} & c_{23} \\
c_{31} & c_{32} & c_{33}
\end{bmatrix}.
$$

131. A square (10x10) matrix $A$ has been factored into the product of a lower (L) and an upper (U) triangular matrix i.e.,

$$
A = LU.
$$

Ignoring additions and subtractions and counting only either multiplications or divisions as operations, roughly how many operations are now required in general to solve an equation of the form:

$$
LUx = b
$$

for a given right hand side vector $b$?

- roughly 10,000
• roughly 1000
• roughly 300
• roughly 100
• roughly 10

Mark the single most appropriate answer.

132. Elementary row-reduction using the Gaussian elimination procedure, (no row interchanges were needed) of the (2x2) symmetric matrix \( A \) produced the upper triangular matrix \( U \) shown below:

\[
U = \begin{bmatrix}
1 & 3 \\
0 & 1
\end{bmatrix}.
\]

(Thus, \( A = LU \) for some triangular matrix \( L \) with 1’s along the diagonal). Find the matrix \( A \).

\[
A = \quad 
\]

133. Let \( S \) be the subset of the \((x,y)\)-plane given by

\[
S = \{(x, y) / y = 3x + 1\}.
\]

Is \( S \) a subspace of the \((x,y)\)-plane?

(i) Yes  
(ii) No

134. For what values of the constants \( a, b \) and \( c \) does the set of equations given below in augmented matrix form have at least one solution?

\[
\begin{bmatrix}
1 & 2 & 1 & a \\
0 & 0 & -1 & b - a \\
0 & 0 & 1 & c - a \\
\end{bmatrix}
\]

Ans..................
135. Let A be an \((mxn)\) matrix. Consider the linear transformation \(f(.) : \mathbb{R}^n \to \mathbb{R}^m\) given by
\[
f(x) = Ax.
\]
(a) What must be the rank of A to ensure that \(f(.)\) is a one-one mapping, i.e., to ensure that for all pairs \(x, y\)
\[
f(x) = f(y) \implies x = y? \]
Ans..........................

(b) What must be the rank of A to ensure that \(f(.)\) is onto, i.e., to ensure that for every \(y\) in \(\mathbb{R}^m\), there is at least one \(x\) in \(\mathbb{R}^n\) such that
\[
f(x) = y? \]
Ans..........................

136. Of the 3 vectors \(a, b\) and \(c\) lying in 3-dimensional Euclidean space (i.e., \(\mathbb{R}^3\)), only \(a\) and \(b\) lie in the \((x,y)\)-plane. Also it is known that the vectors \(a\) and \(b\) do not lie on the same straight line passing through the origin. Do these conditions guarantee that \(a, b\) and \(c\) form a basis for \(\mathbb{R}^3\)?

(i) Yes \quad (ii) No. \quad (iii) cannot say.

137. A is an \((mxn)\) matrix and \(x_1\) and \(x_2\) distinct vectors belonging to the rowspace of A. Is it possible that
\[
A x_1 = A x_2? \]
(Hint: Consider \(A(x_1 - x_2)\).)

(i) Yes \quad (ii) No.

138. Find the angle \(\theta\) between the vectors
\[
x = 1/2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
\]
and

\[
y = \begin{bmatrix}
0 \\
0 \\
0 \\
1 
\end{bmatrix}
\]

lying in \( \mathbb{R}^4 \).

\[\text{Ans.} \]

139. Given

\[
A = LU = \begin{bmatrix}
1 & 0 & 0 & 3 \\
2 & 1 & 0 & 8 \\
1 & 7 & 1 & -5 \\
1 & 7 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & -3 & -1 \\
0 & 1 & 3 \\
0 & 0 & -22 \\
0 & 1 & 0
\end{bmatrix},
\]

solve

\[
Ax = \begin{bmatrix}
3 \\
8 \\
-5
\end{bmatrix}
\]

without multiplying L and U to obtain A first. Show all intermediate steps clearly.

6 pts

140. Determine a basis for and the dimension of the column-space of the matrix A given by:

\[
\begin{bmatrix}
1 & -2 & 2 & -1 \\
-3 & 6 & 1 & 10 \\
1 & -2 & -4 & -7
\end{bmatrix}
\]

6 pts

141. Let \( P(d) \) denote the vector space of all polynomials having real coefficients and of degree \( \leq d \), i.e.,

\[
P(d) = \{ a_0 + a_1X + a_2X^2 + \ldots + a_dX^d / \text{all } a; \text{real} \}.
\]

Since integration of a polynomial of degree \( \leq 3 \) yields a polynomial of degree \( \leq 4 \), the integration operator \( \Psi \) may be viewed as a linear transformation mapping from \( P(3) \) to \( P(4) \).
(a) Write down a matrix $S$ that represents this linear transformation. Identify clearly your choice of basis for $P(3)$ and $P(4)$.

6 pts

(b) What is the nullspace of this matrix?

2 pts

Sp 1991

142. The matrix $A$ as well as it’s inverse $A^{-1}$ are given below:

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 5 & -3 \\ -3 & 2 & -4 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} 14 & -8 & -1 \\ -17 & 10 & 1 \\ -19 & 11 & 1 \end{bmatrix}.$$ 

The matrix

$$B = \begin{bmatrix} 2 & 5 & -3 \\ 1 & 3 & -2 \\ -3 & 2 & -4 \end{bmatrix}$$

is obtained by interchanging the first two rows of $A$. Write down the inverse of $B$.

2 points

ANS

$$B^{-1} =$$

143. In the $n$-dimensional space $\mathbb{R}^n$, let addition and scalar multiplication be defined as follows:

$$x \oplus y = x - y, \quad \text{and} \quad c \cdot x = -cx$$

for any pair of vectors $x, y \in \mathbb{R}^n$ and any real number (scalar) $c$. Under this definition of addition ($\oplus$) and scalar multiplication ($\cdot$), is $\mathbb{R}^n$ a vector space? Justify your answer briefly.

2 points
ANS:

Justification:

144. Determine the dimension of the nullspace of the $(3 \times 4)$ matrix

$$A = \begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & 1 & -5 & -1 \\ 0 & -1 & 5 & 1 \end{bmatrix}.$$

2 points

ANS The dimension of the nullspace is \_\_.

145. If the $n$ columns of an $n \times n$ matrix $A$ are linearly independent, the same is true of the columns of the matrix $A^2$.

2 points

(i) True \hspace{1cm} (ii) False

146. Let $W$ denote the subspace of $(4 \times 4)$ matrices spanned by the elementary matrices $E_{ij}$, with ones on the diagonal and at most one nonzero entry below. What is the dimension of $W$?

2 points

ANS The dimension of $W$ = \_.

147. Let $A$ be an $(m \times n)$ matrix. In least-squares approximation, given an $(m \times 1)$ column vector $\underline{b}$, one determines an unique vector $A\underline{x}$ that minimizes

$$||A\underline{x} - \underline{b}||^2.$$

Is the mapping from $\underline{b}$ to $A\underline{x}$ a linear transformation?
148. The vector space $V$ has a basis containing exactly 4 vectors. Then

(a) any 3 vectors in $V$ are linearly independent,
(b) any 5 vectors span $V$,
(c) any 5 vectors in $V$ are linearly dependent,
(d) None of the above is necessarily true.

149. Exhibit a pair $(\mathbf{x}, \mathbf{y})$, of distinct, non-zero vectors belonging to $\mathbb{R}^3$ satisfying

\[ |\mathbf{x}^T \mathbf{y}| = ||\mathbf{x}|| ||\mathbf{y}||. \]

ANS

\[ \mathbf{x} = \quad \text{and} \quad \mathbf{y} = \quad . \]

150. Let $\mathbf{b}$ be an $(m \times 1)$ column vector and $A$ an $(m \times n)$ matrix. Let $\mathbf{x}$ denote the $(n \times 1)$ vector that minimizes

\[ ||Ax - b||^2. \]

In not more than two lines (you may use equations if you wish) identify precisely, the set of all vectors $\mathbf{b}$ for which the corresponding minimizing vector $\mathbf{x} = \mathbf{0}$.

ANS
151. Write down a basis for $W^\perp$ where $W$ is the nullspace of the matrix

$$A = \begin{bmatrix} 1 & 4 & 6 \\ 0 & 2 & 3 \\ 0 & 2 & 3 \end{bmatrix}.$$  

ANS

A basis is

2 points

152. In this question, we will identify points $(x, y, z)$ in 3-dimensional space $\mathbb{R}^3$ with the vectors $[x, y, z]^t$.

Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ that rotates every point in 3-dimensional space 90° about the z-axis in the direction shown in the figure above.

(a) Write down the transformed coordinates $(a, b, c)$ (i.e., after rotation) of the point $(1, 2, 5)$.

1 point

(b) Determine a matrix that carries out the linear transformation $T$. (Note that the standard basis $\{e_1, e_2, e_3\}$ is used to provide a coordinate representation for both domain as well as range of $T$).

4 points
(c) Is the answer to part (b) unique? 

1 point

11 (a) ANS

\[(a, b, c) = \]

11 (b) ANS

The matrix of the linear transformation is:

11 (c) ANS

Answer to the uniqueness question is:

153. (a) Reduce the matrix

\[ A = \begin{bmatrix} 1 & -2 & 2 & -1 \\ -3 & 6 & 1 & 10 \\ 1 & -2 & -4 & -7 \end{bmatrix} \]

to an (row-reduced) echelon matrix \( U \).

3 points

(b) Hence determine a basis \( B_{\text{row}} \) for the rowspace \( \text{Row}(A) \) as well as a basis \( B_{\text{col}} \) for the column space \( \text{C}(A) \) of \( A \).

3 points

ANS 12(a) The echelon matrix \( U \) is given by:
\[ U = \]

**ANS** 12(b)

\[ B_{row} = \]

\[ B_{col} = \]

154. (a) State in the form of an equation, the orthogonality principle as it applies to least-squares approximation.

2 points

(b) Given the QR decomposition

\[ A = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & 1 \end{bmatrix}, \]

find the vector \( \overline{Ax} \) that minimizes \( \| \overline{Ax} - \overline{b} \| ^2 \) when

\[ \overline{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \]

Explain your approach in a brief sentence.

4 points

13(a) **ANS**

The orthogonality principle:

13(b) **ANS**
My approach is the following:
The minimizing vector $Ax$ is:

$$\text{Sp 1995}$$

155. $A$ is a $(3 \times 3)$ matrix and

$$B = \begin{bmatrix} 4 & 9 & 2 \\ 3 & 5 & 6 \\ 8 & 1 & 7 \end{bmatrix}.$$  

Write down the matrix $A$ if

$$AB = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \begin{bmatrix} 4 & 9 & 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 3 & 5 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} \begin{bmatrix} 8 & 1 & 7 \end{bmatrix}. $$

2 points

156. For what values of the constants $a,b$ and $c$ does the set of equations given below in augmented matrix form have at least one solution?

$$\begin{bmatrix} 3 & 6 & 9 & a \\ 0 & 0 & 2 & b + a \\ 0 & 0 & 6 & c - 2a \end{bmatrix}$$

Ans......................

157. The matrix $A$ as well as its inverse $A^{-1}$ are given below:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix}. $$

The matrix

$$B = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ -6 & 2 & 3 \end{bmatrix}$$

is obtained by interchanging the first two rows of $A$. Write down the inverse of $B$.

2 points

**ANS**
Let $V$ be the set of all positive real numbers, with addition ($\oplus$) and multiplication ($\cdot$) defined by:

- $x \oplus y = xy$
- $c \cdot x = x^c$ where $c$ is any scalar (i.e., any real number).

Is $V$ with these operations, a vector space?

(i) True  (ii) False

159. The collection of all $(3 \times 3)$ matrices forms a vector space. Does the set of all nonsingular $(3 \times 3)$ matrices form a subspace of this vector space?

(i) True  (ii) False

2 points

160. If the $n$ columns of an $n \times n$ matrix $A$ are linearly independent, the same is true of the columns of the matrix $A^2$.

2 points

(i) True  (ii) False

161. Let $W$ denote the subspace of $(4 \times 4)$ matrices spanned by the elementary matrices $E_{ij}$, with ones on the diagonal and at most one nonzero entry below. What is the dimension of $W$?

2 points
The dimension of $W = \ldots$.

162. $A$ is an $(m \times n)$ matrix with $n > m$. Then

(a) the equation $Ax = 0$ always has at least one nonzero solution
(b) the rows of $A$ are linearly dependent
(c) the columns of $A$ are linearly dependent
(d) none of the above

Indicate with a $(\checkmark)$ the statement(s) you think is (are) correct. 2 points

163. In a certain system $Ax = 0$ of homogeneous linear equations involving the four unknowns $x_1, x_2, x_3$ and $x_4$, $A$ is a $(3 \times 4)$ matrix and the basic variables $x_1$ and $x_3$ are related as follows to the free variables $x_2$ and $x_4$:

$$x_1 = 2x_2 + x_4 \quad \text{and} \quad x_3 = 15x_2 - 27x_4.$$  

Write down a basis for the nullspace of the matrix $A$. 2 points

164. Express the matrix

$$A = \begin{bmatrix} 0 & 0 & 4 \\ -6 & 6 & -12 \\ 3 & 0 & 9 \end{bmatrix}$$

in the form $PA = LDW$ where

- $P$ is a $(3 \times 3)$ permutation matrix
- $L$ is a $(3 \times 3)$ lower triangular matrix with 1’s along the diagonal
- $D$ is a diagonal matrix
- $W$ is a $(3 \times 3)$ upper triangular matrix with 1’s along the diagonal

i.e., find the matrices $P$, $L$, $D$ and $W$. 8 points
165. Find a basis for the vector space

\[ W = \langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4 \rangle \]

spanned by the \((4 \times 1)\) vectors

\[
\mathbf{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ -1 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 3 \\ 6 \\ 8 \\ -2 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} -1 \\ -1 \\ -3 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{a}_4 = \begin{bmatrix} -2 \\ -3 \\ -5 \\ 1 \end{bmatrix}.
\]

4 points

166. Does the system of linear equations

\[
\begin{align*}
x_1 - x_2 + 2x_3 &= 1 \\
2x_1 + 2x_3 &= 1 \\
x_1 - 3x_2 + 4x_3 &= 2.
\end{align*}
\]

have a solution? If so, describe explicitly, all solutions. 6 points
167. Let $W_1$ and $W_2$ be subspaces of $\mathbb{R}^m$. Let $\mathcal{B}_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ and $\mathcal{B}_2 = \{\beta_1, \beta_2, \ldots, \beta_l\}$ be a basis for $W_1$ and $W_2$ respectively. You may assume

\[ k \geq l \quad \text{and} \quad m > (k + l). \]

Let $A$ be the $(m \times (k + l))$ matrix given by

\[
\begin{bmatrix}
\alpha_1 & \alpha_2 & \cdots & \alpha_k & \beta_1 & \beta_2 & \cdots & \beta_l
\end{bmatrix}.
\]

In terms of $m, k, l$ what are the maximum and minimum possible dimensions of the nullspace of $A$?

2 points

**ANS** The dimension of the nullspace can range between:

168. The distinct eigenvalues of a $(4 \times 4)$ complex matrix $A$ are $-j, j, +1,$ and $-1$ where as usual, $j = \sqrt{-1}$. What is the least value of the integer $k$ such that

\[ A^k = I_4 \]

where $I_4$ is the $(4 \times 4)$ identity matrix?

2 points

**ANS** least value of $k = \ldots$

169. Let $P$ be an $(n \times n)$ (with $n \geq 2$) projection matrix, i.e., $P$ is a non-zero matrix that satisfies

\[ P = P^T = P^2. \]

Then $P$ has precisely two distinct eigen values, namely 0 and 1.

(i) True \hspace{1cm} (ii) False

2 points
170. Let $a$ be an arbitrary complex number. Let $A$ be the $(2 \times 2)$ matrix

$$
\begin{bmatrix}
a & a^* \\
a^* & a
\end{bmatrix}.
$$

Is $A$ always guaranteed to have a pair of orthonormal vectors as eigenvectors?

(i) True        (ii) False

2 points

171. Let $A$ be a $(3 \times 3)$ matrix whose determinant equals 7. Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be the $(1 \times 3)$ row vectors corresponding to the rows of $A$, i.e.,

$$
A = \begin{bmatrix}
\mathbf{e}_1 \\
\mathbf{e}_2 \\
\mathbf{e}_3
\end{bmatrix}.
$$

What is the determinant of the matrix

$$
B = \begin{bmatrix}
\mathbf{e}_2 + \mathbf{e}_3 \\
\mathbf{e}_1 + \mathbf{e}_3 \\
\mathbf{e}_1 + \mathbf{e}_2
\end{bmatrix}?
$$

ANS $\det(B) =$

2 points

172. Identify any two distinct points lying on the straight line given by the equation:

$$
\begin{vmatrix}
x & y & 4 \\
2 & 8 & 4 \\
1 & 7 & 4
\end{vmatrix} = 0.
$$

2 points

ANS The two points are:

173. The matrix

$$
A = \begin{bmatrix}
1 & -3 \\
-3 & 1
\end{bmatrix}
$$

is
(a) positive definite
(b) positive semidefinite
(c) negative definite
(d) none of the above

2 points

174. Let $A$ be a positive definite Hermitian matrix. Then for every vector $\underline{x}$ having length $||\underline{x}|| = 1$, the value of the quadratic form

$$\underline{x}^H A \underline{x}$$

cannot exceed the trace of the matrix $A$.

(i) True
(ii) False

2 points

175. The diagonal form $\Lambda$ of a certain diagonalizable complex matrix $A$ (i.e., thus for some $S$, $A = S \Lambda S^{-1}$) is of the form:

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \frac{3 + \sqrt{5}}{2} \end{bmatrix},$$

where $j = \sqrt{(-1)}$. Then

(a) $A$ could be a real-valued matrix;
(b) $A$ could be a Hermitian matrix;
(c) $A$ could be a unitary matrix;
(d) none of the above is possible.

2 points

176. Every real, square matrix $A$ is similar to its transpose $A^T$.

(i) True
(ii) False
177. Rewrite the scalar recursion

\[ f_k = 2f_{k-1} - 3f_{k-2} + f_{k-3}, \quad k \geq 3, \]

as a vector recursion of the form

\[ \mathbf{a}_k = A\mathbf{a}_{k-1} \]

for a suitably chosen matrix \( A \) and vector \( \mathbf{a}_k \). Clearly identify the components of both matrix as well as vector.

2 points

178. Consider the differential equation

\[ \frac{du(t)}{dt} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} u(t). \]

Which of the following is then true:

(a) for some initial conditions \( u(0) \), the output \( u(t) \to 0 \) as \( t \to \infty \);
(b) for some initial conditions \( u(0) \), the output \( u(t) \to \infty \) as \( t \to \infty \);
(c) no matter what the initial conditions are, the output will neither decrease to zero nor remain unbounded as \( t \to \infty \);
(d) none of the above

2 points

179. Let \( A \) be the matrix

\[ A = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}. \]

(a) Write down the characteristic polynomial \( f(x) \) of \( A \).

2 points

ANS \( f(x) = \]

(b) What is the minimum polynomial \( p(x) \) of \( A \)?

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(c) Is $A$ diagonalizable? Explain your answer in a single sentence.

ANS

180. The matrices $A$ and $B$

\[
A = \frac{1}{4} \begin{bmatrix}
10 & 2 & 0 & 0 \\
-2 & 6 & 0 & 0 \\
0 & 0 & 10 & 2 \\
0 & 0 & -2 & 6
\end{bmatrix} \quad \text{and} \quad B = \frac{1}{4} \begin{bmatrix}
9 & 1 & 1 & 1 \\
-1 & 7 & -1 & -1 \\
1 & 1 & 9 & 1 \\
-1 & -1 & -1 & 7
\end{bmatrix}
\]

share the same characteristic polynomial

\[
f(x) = (x - 2)^4
\]

as well as the same minimum polynomial

\[
p(x) = (x - 2)^2.
\]

Determine the Jordan canonical forms $J_A$ and $J_B$ of the two matrices.

ANS

\[
J_A = \quad ; \quad J_B = \]

My approach:
181. Given the quadratic

$$3x_1^2 - 2x_1x_2 + 3x_2^2,$$

it is known that there exists a linear change of variables of the form

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

which will allow the quadratic form to be written in the form

$$ay_1^2 + by_2^2$$

for some real numbers $a$ and $b$. Find the matrix $A$ as well as the real numbers $a$ and $b$. Before proceeding to the computations, outline your approach in a brief sentence.

6 points

ANS

$$a = , b = , A =$$

My approach:

extra working space

182. Let

$$A = Q_1 \Sigma Q_2^T$$

be the singular value decomposition of the $(3 \times 4)$ matrix $A$. The matrix $A$ has rank 2 and the matrices in the decomposition are given by

$$Q_1 = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{bmatrix}.$$

In terms of the elements of the matrices $Q_1$, $\Sigma$ and $Q_2$, write down an orthonormal basis for
(a) the columnspace of $A$;  

2 points

(b) the nullspace of $A$.  

2 points

Finally, determine the eigenvalues of $A^t A$.  

1 point

F 1992

183. Let $W_1$ be the set of all polynomials of degree $\leq 6$, whose first 4 coefficients are equal, i.e.,

$$W_1 = \{ c_0 + c_1 X + c_2 X^2 + c_3 X^3 + c_4 X^4 + c_5 X^5 + c_6 X^6 \mid c_0 = c_1 = c_2 = c_3 \}.$$  

Let $W_2$ be the set of all polynomials of degree $\leq 6$, whose last 5 coefficients are equal, i.e.,

$$W_2 = \{ c_0 + c_1 X + c_2 X^2 + c_3 X^3 + c_4 X^4 + c_5 X^5 + c_6 X^6 \mid c_2 = c_3 = c_4 = c_5 = c_6 \}.$$  

(All coefficients are real). Find the dimension of the sumspace $W_1 + W_2$ as a vector space over the real numbers.  

2 points

184. Let $A$ be a $(3 \times 3)$ matrix whose determinant equals 4. Let $r_1, r_2$ and $r_3$ be the $(1 \times 3)$ row vectors corresponding to the rows of $A$, i.e.,

$$A = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}.$$  

Determine the determinant of the matrix

$$B = \begin{bmatrix} r_1 - r_2 \\ r_1 + r_3 \\ r_2 - r_3 \end{bmatrix}.$$  

2 points
185. In this question, the underlying field of scalars is the set of all complex numbers \( \mathbb{C} \). Let \( A \) be a matrix with complex components. As vector spaces over \( \mathbb{C} \), is it true that the rowspace of \( A \) is the orthogonal complement of the nullspace of \( A \), i.e.,

\[
( \text{Row}(A) )^\perp = \mathcal{N}(A)
\]

2 points

(i) True  
(ii) False

186. Again, in this question, the underlying field of scalars is the set of all complex numbers. Then, the distance between any two vectors \( \underline{x}, \underline{y} \in \mathbb{C}^n \), is defined by

\[
d(\underline{x}, \underline{y}) = ||\underline{x} - \underline{y}|| = \sqrt{(\underline{x} - \underline{y})^H(\underline{x} - \underline{y})}.
\]

Does an unitary matrix \( U \) always preserve distances, i.e., is it true that for all \( \underline{x}, \underline{y} \in \mathbb{C}^n \),

\[
d(\underline{x}, \underline{y}) = d(U\underline{x}, U\underline{y}) ?
\]

2 points

(i) True  
(ii) False  
(iii) Cannot Say.

187. Let \( a \) be an arbitrary complex number. Let \( A \) be the \((2 \times 2)\) matrix

\[
\begin{bmatrix}
a & a^* \\
a^* & a
\end{bmatrix}
\]

where \( (\cdot)^* \) denotes the complex conjugate. Is \( A \) a normal matrix?

2 points

(i) True  
(ii) False  
(iii) Cannot Say.

188. The diagonal form of a certain diagonalizable complex matrix \( A \) is given by

\[
A = \begin{bmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 3i + \frac{3}{5}
\end{bmatrix},
\]

where \( i = \sqrt{-1} \). (Thus, \( A = SAS^{-1} \) for some \( S \)). Then

51
(a) $A$ could be a real-valued matrix
(b) $A$ could be a Hermitian matrix
(c) $A$ could be an unitary matrix
(d) None of the above is possible.

2 points

189. Let $P$ be a $(n \times n)$ projection matrix, $P \neq [0]$, $P \neq I$, where $[0]$ and $I$ are the $(n \times n)$ all-zero and identity matrices respectively. Then

(a) $P$ has at most two distinct eigen values and these are 0 and 1,
(b) $P$ has exactly two distinct eigen values and these are 0 and 1,
(c) $P$ has at least two distinct eigen values and these are 0 and 1.

Indicate the single answer that you feel is most appropriate.

2 points

190. $A$, $S$, $C$ and $\Lambda$ are all $(n \times n)$ matrices. $S$ and $C$ are nonsingular and $\Lambda_1$, $\Lambda_2$ are diagonal matrices. It turns out that $A$ can be expressed in the form

$$A = SA_1S^{-1} = C\Lambda_2C^H.$$

Then

(a) the column vectors of $S$ must necessarily be a set of linearly independent eigen vectors of $A$.
(b) the column vectors of $C$ must necessarily be a set of linearly independent eigen vectors for $A$.
(c) none of the above is necessarily true.

2 points

191. Let $U$ be an $(n \times n)$ unitary matrix. Then for every complex vector $\underline{x} \in \mathbb{C}^n$

$$\underline{x}^H U \underline{x}$$

is always real.

(i) True  (ii) False  (iii) Cannot say.

52
192. Let \( \mathbf{x} \) be the \((3 \times 1)\) real vector \([x_1 \; x_2 \; x_3]^T\) and consider the real quadratic form

\[
f(\mathbf{x}) = x_1^2 + 3x_2^2 + 5x_3^2 - x_1x_2 - 8x_2x_3
\]

in the three components of \( \mathbf{x} \). Express the quadratic form as

\[
f(\mathbf{x}) = \mathbf{x}^T \mathbf{A} \mathbf{x}
\]

for a suitably chosen symmetric matrix \( \mathbf{A} \) and identify \( \mathbf{A} \).

2 points

193. In this question, we will identify points \((x, y, z)\) in 3-dimensional space \( \mathbb{R}^3 \) with the vectors \([x \; y \; z]^T\).

(a) Let \( \mathcal{A} \) and \( \mathcal{B} \) be two different bases for \( \mathbb{R}^3 \) given by

\[
\mathcal{A} = \begin{Bmatrix} 
\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{Bmatrix}
\]

and

\[
\mathcal{B} = \begin{Bmatrix} 
\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \end{Bmatrix}.
\]

Let the matrices \([I]_{AB}\) and \([I]_{BA}\) correspond to a change of coordinates from \( \mathcal{A} \) to \( \mathcal{B} \) and vice-versa, i.e.,

\[
[I]_{AB} \ [x]_{A} = [x]_{B}
\]

and

\[
[I]_{BA} \ [x]_{B} = [x]_{A}
\]

for all vectors \( \mathbf{x} \) belonging to \( \mathbb{R}^3 \). Determine either \([I]_{AB}\) or \([I]_{BA}\), but make clear in your working, as to which of the two matrices you have chosen to determine.

4 points

(b) Consider the linear transformation \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) that rotates every point in 3-dimensional space \( 90^\circ \) about the z-axis in the direction shown in the figure above. Write down the transformed coordinates \((a, b, c)\) (i.e., after rotation) of the point \((1, 0, 0)\).

1 point
(c) Determine the matrix $[T]_B^B$ that carries out the linear transformation when the vectors in both the domain and range of the transformation $T$ are represented with respect to the basis $B$, i.e.,

$$[T]_B^B : [x]_B \rightarrow [T(x)]_B$$

for all vectors $x$ in $\mathbb{R}^3$, where $T(x)$ denotes the image of $x$ under the linear transformation $T$. Show all of your working clearly. (See previous page for the basis $B$).

6 points

194. Let $A$ and $B$ be the matrices

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$  

Determine for each matrix, whether or not the matrix is diagonalizable. Show clearly, your working.

5 points

195. Find the determinants of the upper-left submatrices of the matrix $A$ given by

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}.$$  

Is $A$ positive-definite?

4 points

196. Let $a > b > c > 0$ be real numbers. Show that the minimum value of

$$ay_1^2 + by_2^2 + cy_3^2$$

subject to

$$y_1^2 + y_2^2 + y_3^2 = 1$$

where $y_1, y_2$ and $y_3$ are real numbers, equals $c$.

2 points
197. What is the minimum value of the quadratic form
\[ \underline{x}^T A \underline{x} \] where \[ A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \]
subject to \( \underline{x}^T \underline{x} = x_1^2 + x_2^2 + x_3^2 = 1 \), where \( x_1, x_2 \) and \( x_3 \) are real numbers. Explain your working clearly. (Hint: Use \( A = Q\Lambda Q^T \), for some orthogonal matrix \( Q \). Note that \( Q \) preserves lengths).

5 points

Sp 1993

198. Let \( Q \) be an \( (n \times n) \) real, orthogonal matrix. Then for every vector \( \underline{x} \in \mathbb{R}^n \),
\[ \|Q\underline{x}\| = \|\underline{x}\|. \]

(i) True \hspace{1cm} (ii) False

Let \( A \) be an \( (n \times n) \) matrix that satisfies
\[ \|A\underline{x}\| = \|\underline{x}\| \]
for every vector \( \underline{x} \in \mathbb{R}^n \). Then \( A \) must be an orthogonal matrix.

(i) True \hspace{1cm} (ii) False

2 points

199. Let \( A \) be an \( (m \times n) \), real matrix. Let \( \underline{b} \) be an arbitrary vector in \( \mathbb{R}^m \). Consider the set
\[ S = \{ A\underline{x} \mid \underline{x} \text{ is a solution to the equation } A^T A \underline{x} = A^T \underline{b} \}. \]

How many distinct elements does \( S \) contain?

(a) \( S \) always contains just one distinct element
(b) \( S \) always contains an infinite number of distinct elements
(c) the number of distinct elements depends upon the particular \( \underline{b} \)
(d) none of the above is necessarily true

2 points

200. The vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ in $\mathbb{R}^3$ are orthogonal. The projection of a certain vector $\mathbf{b}$ (also in $\mathbb{R}^3$) onto $\mathbf{a}_1$ turns out to be the vector

$$\begin{bmatrix}
4 \\
1 \\
2
\end{bmatrix}.$$ 

The projection of $\mathbf{b}$ onto the subspace $W = \langle \mathbf{a}_1, \mathbf{a}_2 \rangle$ of $\mathbb{R}^3$ spanned by both $\mathbf{a}_1$ and $\mathbf{a}_2$ turns out to be the vector

$$\begin{bmatrix}
4 \\
2 \\
0
\end{bmatrix}.$$ 

Determine the projection of $\mathbf{b}$ onto $\mathbf{a}_2$.

2 points

201. Let $\mathcal{A} = \{ \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \}$ and $\mathcal{B} = \{ \mathbf{b}_1, \mathbf{b}_2 \}$ be bases for the orthogonal subspaces $W_A$ and $W_B$ respectively. $W_A$ and $W_B$ are subspaces of $\mathbb{R}^6$. How is the columnspace of the matrix

$$A = \begin{bmatrix}
\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{b}_1 & \mathbf{b}_2
\end{bmatrix}$$

related to $W_A$ and $W_B$? What is the rank of $A$?

2 points

202. $A$ is a $(3 \times 3)$ matrix whose $(i,j)^{th}$ element is denoted as usual by $a_{i,j}$. Let $A_{i,j}$ denote the cofactor of $A$ associated with $a_{i,j}$. Find a $(3 \times 3)$ matrix $B$ whose determinant has the following cofactor expansion

$$\det(B) = xA_{13} + yA_{23} + zA_{33}$$

2 points
203. The matrix $A$ is a $(4 \times 4)$ real matrix having only two distinct eigen values, namely, 3 and 4. What are the possible values of the determinant of $A$?  

2 points

204. The eigenvalues of a certain complex $(4 \times 4)$ matrix $A$ are 

$$1, j, -1, -j$$

where as usual, $j = \sqrt{-1}$. What is the least value of the integer $n$ such that 

$$A^n = I,$$

where $I$ is the $(4 \times 4)$ identity matrix?  

2 points

205. Consider the sequence of integers that satisfy the recursion 

$$G_k = G_{k-1} - 2G_{k-2} + G_{k-3}, \quad \forall k \geq 3.$$ 

The initial conditions are $G_0 = 1$, $G_1 = 2$ and $G_2 = 4$. Rewrite this scalar recursion as a single-step $(3 \times 1)$ vector recursion.  

2 points

206. Let $A$ be an $(n \times n)$ Hermitian matrix. Then for every arbitrary complex vector $\mathbf{z} \in \mathbb{C}^n$ (where $\mathbb{C}$ denotes the set of all complex numbers), $\mathbf{z}^H A \mathbf{z}$ is always real.

(i) True  
(ii) False  

2 points

207. Let $W$ be the subspace of $\mathbb{R}^3$ spanned by the orthogonal vectors 

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

Express the vector 

$$\mathbf{z} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

as the sum of two vectors $\mathbf{z}_1$ and $\mathbf{z}_2$, where $\mathbf{z}_1 \in W$ and $\mathbf{z}_2$ is perpendicular to all the vectors in $W$.  

Explain your intended procedure clearly before you begin.
208. Let $\mathcal{A}$ and $\mathcal{B}$ denote the two bases for $\mathbb{R}^2$ given by

$$\mathcal{A} = \{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \} \quad \text{and} \quad \mathcal{B} = \{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \}.$$ 

Note that $\mathcal{A}$ is the standard basis for $\mathbb{R}^2$. Let $S$ be the matrix

$$S = \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}.$$ 

If coordinate representation with respect to the basis $\mathcal{B}$ is used instead (of the standard basis $\mathcal{A}$), determine the representation $S_{\mathcal{B}}$ of $S$ with respect to the basis $\mathcal{B}$.

8 points

(As intermediate steps, determine and identify clearly, the change-of-basis matrices $T_{\mathcal{A}\mathcal{B}}$ (from $\mathcal{A} \rightarrow \mathcal{B}$) and $T_{\mathcal{B}\mathcal{A}}$ (from $\mathcal{B} \rightarrow \mathcal{A}$).)

209. Find the limiting values of $y_k$ and $z_k$ (i.e., the values as $k \rightarrow \infty$) if

$$y_{k+1} = 0.8y_k + 0.3z_k \quad y_0 = 0 \quad \text{(1)}$$

$$z_{k+1} = 0.2y_k + 0.7z_k \quad z_0 = 5 \quad \text{(2)}$$

8 points

Fall 1993

210. Let $W_1$ be the set of all polynomials of degree $\leq 6$, whose first 4 coefficients are equal, i.e.,

$$W_1 = \{ c_0 + c_1X + c_2X^2 + c_3X^3 + c_4X^4 + c_5X^5 + c_6X^6 \mid c_0 = c_1 = c_2 = c_3 \}.$$ 

Let $W_2$ be the set of all polynomials of degree $\leq 6$, whose last 5 coefficients are equal, i.e.,

$$W_2 = \{ c_0 + c_1X + c_2X^2 + c_3X^3 + c_4X^4 + c_5X^5 + c_6X^6 \mid c_2 = c_3 = c_4 = c_5 = c_6 \}.$$ 

(All coefficients are real). Find the dimension of the sumspace $W_1 + W_2$ as a vector space over the real numbers.

2 points
211. Let $A$ be a $(3 \times 3)$ matrix whose determinant equals 4. Let $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ be the $(1 \times 3)$ row vectors corresponding to the rows of $A$, i.e.,

$$A = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{bmatrix}.$$ 

Determine the determinant of the matrix

$$B = \begin{bmatrix} \mathbf{r}_1 - \mathbf{r}_2 \\ \mathbf{r}_1 + \mathbf{r}_3 \\ \mathbf{r}_2 - \mathbf{r}_3 \end{bmatrix}.$$ 

2 points

212. In this question, the underlying field of scalars is the set of all complex numbers $\mathbb{C}$. Let $A$ be a matrix with complex components. As vector spaces over $\mathbb{C}$, is it true that the rowspace of $A$ is the orthogonal complement of the nullspace of $A$, i.e.,

$$(\text{Row}(A))^\perp = \mathcal{N}(A) ?$$

2 points

(i) True     (ii) False

213. Again, in this question, the underlying field of scalars is the set of all complex numbers. Then, the distance between any two vectors $\mathbf{x}$, $\mathbf{y} \in \mathbb{C}^n$, is defined by

$$d(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}|| = \sqrt{(\mathbf{x} - \mathbf{y})^H(\mathbf{x} - \mathbf{y})}.$$ 

Does an unitary matrix $U$ always preserve distances, i.e., is it true that for all $\mathbf{x}$, $\mathbf{y} \in \mathbb{C}^n$,

$$d(\mathbf{x}, \mathbf{y}) = d(U\mathbf{x}, U\mathbf{y}) ?$$

2 points

(i) True     (ii) False     (iii) Cannot Say.

214. Let $a$ be an arbitrary complex number. Let $A$ be the $(2 \times 2)$ matrix

$$\begin{bmatrix} a & a^* \\ a^* & a \end{bmatrix}$$

where $(\cdot)^*$ denotes the complex conjugate. Is $A$ a normal matrix?
215. The diagonal form of a certain diagonalizable complex matrix $A$ is given by

$$
A = \begin{bmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & \frac{3+i4}{5}
\end{bmatrix},
$$

where $i = \sqrt{-1}$. (Thus, $A = SAS^{-1}$ for some $S$). Then

(a) $A$ could be a real-valued matrix
(b) $A$ could be a Hermitian matrix
(c) $A$ could be an unitary matrix
(d) None of the above is possible.

2 points

216. Let $P$ be a $(n \times n)$ projection matrix, $P \neq [0], P \neq I$, where $[0]$ and $I$ are the $(n \times n)$ all-zero and identity matrices respectively. Then

(a) $P$ has at most two distinct eigen values and these are 0 and 1,
(b) $P$ has exactly two distinct eigen values and these are 0 and 1,
(c) $P$ has at least two distinct eigen values and these are 0 and 1.

Indicate the single answer that you feel is most appropriate.

2 points

217. $A$, $S$, $C$ and $\Lambda$ are all $(n \times n)$ matrices. $S$ and $C$ are nonsingular and $\Lambda_1$, $\Lambda_2$ are diagonal matrices. It turns out that $A$ can be expressed in the form

$$
A = SA_1S^{-1} = CA_2C^H.
$$

Then

(a) the column vectors of $S$ must necessarily be a set of linearly independent eigen vectors of $A$.
(b) the column vectors of $C$ must necessarily be a set of linearly independent eigen vectors for $A$. 

2 points
(c) none of the above is necessarily true.

2 points

218. Let \( U \) be an \((n \times n)\) unitary matrix. Then for every complex vector \( x \in \mathbb{C}^n \)

\[
x^H U x
\]

is always real.

(i) True  
(ii) False  
(iii) Cannot say.

2 points

219. Let \( x \) be the \((3 \times 1)\) real vector \([x_1 \ x_2 \ x_3]^T\) and consider the real quadratic form

\[
f(x) = x_1^2 + 3x_2^2 + 5x_3^2 - x_1 x_2 - 8x_2 x_3
\]

in the three components of \( x \). Express the quadratic form as

\[
f(x) = x^T A x
\]

for a suitably chosen symmetric matrix \( A \) and identify \( A \).

2 points

**Sp 1994**

220. Let \( W_1 \) be the set of all polynomials of degree \( \leq 6 \), whose first 3 coefficients are equal, i.e.,

\[
W_1 = \{ c_0 + c_1 X + c_2 X^2 + c_3 X^3 + c_4 X^4 + c_5 X^5 + c_6 X^6 \mid c_0 = c_1 = c_2 \}.
\]

Let \( W_2 \) be the set of all polynomials of degree \( \leq 6 \), whose last 5 coefficients are equal, i.e.,

\[
W_2 = \{ c_0 + c_1 X + c_2 X^2 + c_3 X^3 + c_4 X^4 + c_5 X^5 + c_6 X^6 \mid c_2 = c_3 = c_4 = c_5 = c_6 \}.
\]

(All coefficients are real). Find the dimension of the sumspace \( W_1 + W_2 \) as a vector space over the real numbers.

2 points
221. The vectors

\[
x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad x_4 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix},
\]

are known to form a set of pairwise, orthogonal vectors.

Find the coefficient \( c_3 \) in the coordinate representation

\[
\begin{bmatrix} 10 \\ 4 \\ 0 \\ -6 \end{bmatrix} = c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4.
\]

\[\text{ANS} \quad c_3 = \quad 2 \text{ points}\]

222. In a certain least-squares approximation problem, \( Ax = b \), where \( A \) is a real \( m \times n \) matrix, the solution \( Ax \) turned out to be the zero vector \( \mathbf{0} \). Can you conclude that \( b \) must have been orthogonal to the columnspace of \( A \), i.e., that necessarily \( A^T b = \mathbf{0} \) ?

\( \quad \text{(i) True} \quad \text{(ii) False} \quad \)

\[\text{2 points}\]

223. Let \( A \) be a \( (3 \times 3) \) matrix whose determinant equals 4. Let \( r_1, r_2 \) and \( r_3 \) be the \( (1 \times 3) \) row vectors corresponding to the rows of \( A \), i.e.,

\[
A = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}.
\]

Determine the determinant of the matrix

\[
B = \begin{bmatrix} r_1 - r_3 \\ r_2 - r_2 \\ r_2 + r_3 \end{bmatrix}.
\]

\[\text{2 points}\]

224. An \( (n \times n) \) matrix \( A \) is diagonalizable if and only if all of its \( n \) eigenvalues are distinct.

\[\text{2 points}\]
(i) True  (ii) False

225. In this question, the underlying field of scalars is the set of all complex numbers \( \mathbb{C} \). Let \( A \) be a matrix with complex components. As vector spaces over \( \mathbb{C} \), is it true that the rowspace of \( A \) is the orthogonal complement of the nullspace of \( A \), i.e., is it true that

\[
( \text{Row}(A) )^\perp = \mathcal{N}(A)
\]

2 points

(i) True  (ii) False

226. The diagonal form of a certain diagonalizable complex matrix \( A \) is given by

\[
A = \begin{bmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 3 + 4i
\end{bmatrix},
\]

where \( i = \sqrt{-1} \). (Thus, \( A = SAS^{-1} \) for some \( S \)). Then

(a) \( A \) could be a real-valued matrix
(b) \( A \) could be a Hermitian matrix
(c) \( A \) could be an unitary matrix
(d) None of the above is possible.

2 points

227. Is the matrix

\[
A = \begin{bmatrix}
\cos(t) & \sin(t) \\
-\sin(t) & \cos(t)
\end{bmatrix}
\]

a normal matrix?

2 points

(i) True  (ii) False
228. It is desired to find the real coefficients $C, D, E$ that minimize

$$\sum_{i=1}^{4} [y_i - (C + Dt_i + Et_i^2)]^2$$

where

$$y_1 = 2, \quad t_1 = -1 \quad y_2 = 0, \quad t_2 = 0 \quad y_3 = -3, \quad t_3 = 1 \quad y_4 = -5, \quad t_4 = 2$$

Formulate this problem as a least-squares approximation problem $Ax = b$, i.e., identify the matrix $A$ and the vector $b$.

4 points

229. Apply the Gram-Schmidt orthogonalization process to find an orthonormal basis for the space spanned by the three vectors

$$a = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad c = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$ 

Show your working clearly.

6 points

230. Find the determinant of the $(n \times n)$ matrix $M$ obtained when a vector $[x_1, x_2, \ldots, x_n]^T$ replaces the $j$th column of the $(n \times n)$ identity matrix $I_n$, i.e., $M$ is the matrix

$$M = \begin{bmatrix} 1 & x_1 \\ 1 & \cdot \\ x_j & \cdot \\ \cdot & 1 \\ x_n & 1 \end{bmatrix}$$

4 points

231. Let $A$ be the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$ 

Is $A$ diagonalizable? Explain your answer and show clearly, your working.

4 points

232. There are three major centers for Move-It-Yourself trucks. Every month half of those in Boston and in Los Angeles go to Chicago, the other half stay where they are, and the trucks in Chicago are equally split between Boston and Los Angeles.
Set up a difference equation that represents this recursion and identify the matrix $A$ of the recursion. If initially, all three centers have the same number of trucks, what will be the distribution of trucks after, many, many months?

Show your working clearly.

8 points

233. Find $e^{At}$ if

$$A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$  

6 points

234. (a) The $(4 \times 4)$ matrix $A$ is real, symmetric and unitary. State all that you can deduce regarding the eigenvalues of $A$.

2 points

What are the possible values of the trace of $A$?

2 points

Sp 1995

235. The vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ in $\mathbb{R}^3$ are orthogonal. The projection of a certain vector $\mathbf{b}$ (also in $\mathbb{R}^3$) onto $\mathbf{a}_1$ turns out to be the vector

$$\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}.$$  

The projection of $\mathbf{b}$ onto the subspace

$$W = \langle \mathbf{a}_1, \mathbf{a}_2 \rangle$$

of $\mathbb{R}^3$ spanned by both $\mathbf{a}_1$ and $\mathbf{a}_2$ turns out to be the vector

$$\begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}.$$  

Determine the projection of $\mathbf{b}$ onto $\mathbf{a}_2$.

2 points
236. A is a $(3 \times 3)$ matrix whose $(i, j)^{th}$ element is denoted as usual by $a_{i,j}$. Let $A_{i,j}$ denote the cofactor of $A$ associated with $a_{i,j}$. Find a $(3 \times 3)$ matrix $B$ whose determinant has the following cofactor expansion

$$\det(B) = xA_{13} + yA_{23} + zA_{33}$$

2 points

237. Identify any two distinct points lying on the straight line given by the equation:

$$\begin{vmatrix} x & y & 4 \\ 2 & 8 & 4 \\ 1 & 7 & 4 \end{vmatrix} = 0.$$  

2 points

**ANS** The two points are:

238. The matrix $A$ is a $(4 \times 4)$ real matrix having only two distinct eigen values, namely, 3 and 4. What are the possible values of the determinant of $A$?

2 points

239. Rewrite the scalar recursion

$$f_k = 2f_{k-1} - 3f_{k-2} + f_{k-3}, \quad k \geq 3,$$

as a vector recursion of the form

$$a_k = Aa_{k-1}$$

for a suitably chosen $(3 \times 3)$ matrix $A$ and vector $a_k$. Clearly identify the components of both matrix as well as vector.

2 points

240. Consider the differential equation

$$\frac{du(t)}{dt} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} u(t).$$

Which of the following is then true:

(a) for some initial conditions $u(0)$, the output $u(t) \to 0$ as $t \to \infty;
(b) for some initial conditions \( u(0) \), the output \( u(t) \to \infty \) as \( t \to \infty \);
(c) no matter what the initial conditions are, the output will neither decrease to zero nor remain unbounded as \( t \to \infty \);
(d) none of the above

2 points

241. \( A, S, C \) and \( \Lambda \) are all \((n \times n)\) matrices. \( S \) and \( C \) are nonsingular and \( \Lambda_1, \Lambda_2 \) are diagonal matrices. It turns out that \( A \) can be expressed in the form

\[
A = S\Lambda_1 S^{-1} = C\Lambda_2 C^H.
\]

Then

(a) the column vectors of \( S \) must necessarily be a set of linearly independent eigen vectors of \( A \).
(b) the column vectors of \( C \) must necessarily be a set of linearly independent eigen vectors for \( A \).
(c) none of the above is necessarily true.

2 points

242. Apply the Gram-Schmidt orthogonalization process and derive an orthonormal basis for \( \mathbb{R}^3 \), starting with the three vectors

\[
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}, \quad \begin{bmatrix}
0 \\
1 \\
1
\end{bmatrix}, \quad \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}.
\]

7 points

243. Let \( A \) and \( B \) be the matrices

\[
A = \begin{bmatrix}
3 & -1 \\
1 & 1
\end{bmatrix}
\]

and

\[
B = \begin{bmatrix}
2 & 1 & 2 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}.
\]

Determine for each matrix, whether or not the matrix is diagonalizable. Show clearly, your working.
244. Find the limiting values of $y_k$ and $z_k$ (i.e., the values as $k \to \infty$) if

\begin{align*}
y_{k+1} &= 0.8y_k + 0.3z_k \quad y_0 = 0 \\
z_{k+1} &= 0.2y_k + 0.7z_k \quad z_0 = 5
\end{align*} \hspace{1cm} (3) \hspace{1cm} (4)

8 points

245. The determinant of a square matrix $A$ having complex-valued entries equals $3 + i4$. What is the determinant of $A^H$?

2 points

246. Let $A$ be an $(m \times n)$ matrix and $\underline{b}$ be an $(m \times 1)$ vector.

(a) Under what conditions on $A$ does the equation $A^tAx = A^t\underline{b}$ have a solution $\underline{x}$?

(b) Under what conditions on $A$ does the equation $A^tAx = A^t\underline{b}$ have a unique solution $\underline{x}$ (given that a solution exists)?

2 points

247. Let $P$ denote the plane in $\mathbb{R}^3$ containing the vectors $\underline{x} = [1 \ 1 \ 1]^t$ and $\underline{y} = [1 \ 2 \ 3]^t$. Find a vector $\underline{z}$ in $P$ that is orthogonal to $\underline{x}$.

2 points

248. Assuming that $a, b, c, d$ are all distinct, what is the determinant of the matrix

$$A = \begin{bmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{bmatrix}$$

2 points

249. Determine the eigenvalues of a $(2 \times 2)$ matrix $A$ whose trace equals 9 and whose determinant equals 14.

2 points
250. The $(3 \times 3)$ matrix $A$ has eigenvalues 2, 2, 3. The nullspaces of $A - 2I$ and $A - 3I$ are of dimension 2 and 1 respectively. Then $A$ is diagonalizable.

(i) True  (ii) False  (iii) Cannot say

2 points

251. If $\underline{x}$ and $\underline{y}$ are any two eigenvectors of the matrix $A$, then any linear combination $cx + dy$ of the two eigenvectors will always be an eigenvector of $A$.

(i) True  (ii) False  (iii) Cannot say

2 points

252. The matrix $A$ satisfies the condition $A = -A^H$. Then $A$ is a normal matrix.

2 points

253. Let $A$ be a complex-valued matrix with at least one component that is not real. Then every vector in the nullspace of $A$ is orthogonal to every vector in the rowspace of $A$.

(i) True  (ii) False  (iii) Cannot say

2 points

254. Let $A$ be a $(3 \times 3)$ complex-valued, diagonalizable matrix. Under what conditions on the eigenvalues of $A$ will the vector $\underline{x}(t)$ satisfying

$$\frac{d}{dt} (\underline{x}(t)) = A\underline{x}$$

go to $\underline{0}$ as $t \to \infty$ regardless of the value of $\underline{x}(0)$?

2 points

255. Compute $e^A e^B$ if $A$ and $B$ are square matrices satisfying $A^2 = B^3 = [0]$, where $[0]$ is the all-zero matrix.

2 points
The vectors $\mathbf{x}$ and $\mathbf{y}$ are complex-valued vectors satisfying
\[ \|\mathbf{x} + \mathbf{y}\|^2 = 9 \quad \text{and} \quad \|\mathbf{x} + i\mathbf{y}\|^2 = 15. \]
Compute $\mathbf{x}^H \mathbf{y}$. 

4 points

The result of projecting the vector $\mathbf{z} = [2 \ 6 \ 1]^t$ onto the plane $Q$ in $\mathbb{R}^3$ is the vector $\mathbf{z} = [1 \ 5 \ 3]^t$. Identify as clearly and as explicitly as you can, the plane $Q$. 

4 points

The symmetric matrix
\[ A = \begin{bmatrix} 17 & 8 & 4 \\ 8 & 17 & -4 \\ -4 & -4 & 11 \end{bmatrix} \]
has the eigenvalues 9, 9 and 27. The vector $[2 \ 2 \ -1]^t$ is an eigenvector of $A$ corresponding to eigenvalue 27. Find an orthonormal set of eigenvectors for the matrix $A$. 

6 points

Each number $G_k$, $k \geq 2$, is the average of the two preceding numbers, i.e.,
\[ G_k = \frac{G_{k-1} + G_{k-2}}{2}. \]
Let the initial conditions be $G_0 = 5$, and $G_1 = 10$. Write down a 1-step vector difference equation corresponding to this two-step scalar difference equation. Use this vector form to find a closed-form expression for $G_k$. What is the value of $G_k$ as $k$ tends to $\infty$? 

8 points

Consider the differential equation
\[ y'' + y' - 6y = 0. \]
Set up the corresponding first-order vector differential equation. Solve the differential equation by expressing $y(t)$ in terms of $y(0)$ and $y'(0)$. (Do not simplify your answer any more than you have to). 

6 points

Sp 1998
261. If no plane passing thru the origin in \( \mathbb{R}^3 \) contains a given set of three vectors in \( \mathbb{R}^3 \), then those three vectors form a basis for \( \mathbb{R}^3 \).

(i) True  (ii) False

2 points

262. Let \( V \) be the vector space consisting of all \((4 \times 4)\) matrices. Let \( W \) be the subspace of \( V \) consisting of all \textit{symmetric} \((4 \times 4)\) matrices. What is the dimension of \( W \)?

2 points

263. A rectangular \((m \times n)\) matrix \( A \) has a left inverse \( B \), i.e., an \((n \times m)\) matrix \( B \) such that

\[
BA = I_n
\]

where \( I_n \) is the \((n \times n)\) identity matrix. Which of the following statements is always true?

(a) the rank of \( A \) is \( m \)
(b) the rank of \( A \) is \( n \)
(c) for any \((m \times 1)\) vector \( \vec{b} \), the equation \( A\vec{x} = \vec{b} \) always has at least one solution
(d) if the vector \( \vec{b} \) is such that the equation \( A\vec{x} = \vec{b} \) has a solution, then that solution is the unique solution
(e) none of the above is necessarily true

2 points

264. (a) Let \( S \) and \( T \) be two subspaces of the vector space \( V \). How are the dimensions of \( S, T, S + T \) and \( S \cap T \) related?

1 point

(b) If \( V \) has dimension 13 and \( S \) and \( T \) have dimensions 7 and 8 respectively, what is the smallest possible dimension of \( S \cap T \)?

1 point

265. Does the Pythagoras theorem hold in \( n \)-dimensional space \( \mathbb{R}^n \)? Explain your answer, beginning with a mathematical statement of the Pythagoras theorem in the spaces \( \mathbb{R}^n \).

2 points

266. Let \( i = \sqrt{-1} \). Find the projection of the vector \([1+i, \quad 1-i]^T\) onto the vector \([2, \quad 3i]^T\).

2 points
267. Find the determinant of the (4 × 4) matrix \( A = [a_{i,j}] \), where \( a_{i,j} = i + j, \ 1 \leq i, j \leq 4 \).

2 points

268. Find a basis for the orthogonal complement of the nullspace of the matrix

\[
A = \begin{bmatrix}
1 & 2 & -5 \\
1 & 6 & 5
\end{bmatrix}
\]

2 points

269. Find a basis for the plane thru the origin in \( \mathbb{R}^3 \) consisting of all points (vectors) \((x, y, z)\) satisfying \( x + 2y + 3z = 0 \).

2 points

270. The sequence \( \{a_k, k \geq 0\} \) satisfies the linear recursion

\[
a_k = 2a_{k-1} - 3a_{k-2} + a_{k-3} \text{ for } k \geq 3.
\]

Also, \( a_0 = 0, a_1 = 1 \) and \( a_2 = 2 \). Convert this scalar recursion equation to a vector recursion of the form

\[
u_k = A\nu_{k-1},
\]

just as was done in class for the Fibonacci sequence. In other words, clearly identify the size and components of the matrix \( A \) and relate the components of the vector \( \nu_k \) to elements of the sequence \( \{a_k\} \). (It is not necessary to solve for the values of \( a_k \)).

2 points

271. The vectors

\[
\mathbf{s}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \ \mathbf{s}_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \ \text{and} \ \mathbf{s}_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix},
\]

are pairwise orthogonal. Express the vector \( \mathbf{x} = [1 \ 1 \ 1]^T \) as a linear combination

\[
\mathbf{x} = c_1\mathbf{s}_1 + c_2\mathbf{s}_2 + c_3\mathbf{s}_3
\]

of the \( \mathbf{s}_i \).

4 points

272. The cross product of the real vectors

\[
\mathbf{a} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \ \text{and} \ \mathbf{b} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix},
\]
is the real vector
\[ \mathbf{a} = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} \]
where the components \( \{x_3, y_3, z_3\} \) are found from the determinant below and its expansion in terms of the elements of the first row:
\[ \det \begin{bmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix} = i(x_3) + j(y_3) + k(z_3). \]

Explain as clearly as you can, in not more than three sentences, the reason why the cross-product yields a vector \([x_3 \ y_3 \ z_3]^T\) that is orthogonal to both vectors \([x_1 \ y_1 \ z_1]^T\) and \([x_2 \ y_2 \ z_2]^T\).

4 points

273. Consider the following two basis for the vector space \( P_1 \) of all polynomials of degree less than or equal to one:
\[ \mathcal{A} = \{1, x\} \]
and
\[ \mathcal{B} = \{1 + 2x, 2 - 3x\}. \]
For any polynomial \( f(x) = f_0 + f_1 x \), let \([f]_\mathcal{A}\) and \([f]_\mathcal{B}\) be the coordinate representations of the polynomial \( f(x) \) with respect to the basis \( \mathcal{A} \) and \( \mathcal{B} \) respectively. Thus \([f]_\mathcal{A}\) and \([f]_\mathcal{B}\) are \((2 \times 1)\) vectors. Find a matrix \( T \) such that
\[ [f]_\mathcal{B} = T[f]_\mathcal{A} \]
for all polynomials \( f(x) \in P_1 \). Show all your working clearly.

6 points

274. A student was given three nonzero vectors \( \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\} \) and was asked to use the Gram-Schmidt (G-S) orthogonalization process to obtain from the \( \{\mathbf{a}_i\} \), a set of pairwise orthogonal vectors \( \mathbf{p}_1, \mathbf{p}_2 \) and \( \mathbf{p}_3 \). The student set \( \mathbf{p}_1 = \mathbf{a}_1 \) and then found a nonzero vector \( \mathbf{p}_2 \) by correctly following the G-S process. Thus \( \mathbf{p}_1 \) and \( \mathbf{p}_2 \) were orthogonal and the pair \( \{\mathbf{p}_1, \mathbf{p}_2\} \), spanned the same space spanned by the vectors \( \{\mathbf{a}_1, \mathbf{a}_2\} \).
But when the student computed \( \mathbf{p}_3 \) he found (his method and calculations were correct) to his surprise, that \( \mathbf{p}_3 = \mathbf{0} \).
Identify the conditions on the vectors \( \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\} \) under which this will always happen. Be as precise and as brief as you can.

4 points
275. Is the matrix

\[
A = \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{bmatrix}
\]

diagonalizable? Before you begin, define what is meant by a diagonalizable matrix. Next, determine the eigenvalues and the eigenvectors of \(A\). Finally, explain as clearly as you can, your answer to the question.

8 points

276. Solve the least-squares approximation problem \(Ax \cong b\), where

\[
A = \begin{bmatrix}
1 & 1 \\
1 & -1 \\
1 & 1 \\
1 & -1 \\
\end{bmatrix}
\]

and \(b = [2 \ 2 \ 3 \ 4]^T\). In other words, find a vector \(z\) of the form \(Ax\) such that the mean-square error \(\|b - z\|^2\) is minimized.

4 points

END